Gröbner Bases for Toric Ideals of Acyclic Directed Graphs and their Applications to Minimum Cost Flow Problem

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Abstract

Applications of Gröbner bases to some computationally hard problems in combinatorics using the discreteness of toric ideals have been studied in recent years. On the other hand, the properties of graphs may give insight into Gröbner bases. In this paper, we analyze toric ideals of acyclic directed graphs, which are the most fundamental directed graphs. We focus especially on the number of elements of their reduced Gröbner bases. We show that there exist term orders for which reduced Gröbner bases remain in polynomial order by characterizing the bases in terms of circuits of graphs. We next analyze the number of elements of reduced Gröbner bases with respect to various term orders. We finally discuss applications to the minimum cost flow problem.

1 Introduction

Recently, many algebraic algorithms using *Gröbner bases* have been studied in many fields. Related to some combinatorial problems in graph theory, Gröbner bases for toric ideals of graphs and their application have been studied. De Loera, Sturmfels and Thomas [6] studied the toric ideals of undirected complete graphs, and applied them to the triangulations of second hypersimplices and perfect f-matching problems. Diaconis and Sturmfels [7] studied the toric ideals of bipartite graphs, and applied for sampling from conditional distributions and transportation problems. From the viewpoint of commutative algebra, Ohsugi and Hibi [14] studied the toric ideals of undirected graphs, and showed the conditions when the toric ideals are generated by quadratic binomials. Conversely, the properties of graphs may give insight into Gröbner bases.

We are interested in the Gröbner bases of directed graphs. In this paper, we study the toric ideals of acyclic directed graphs, which are the most fundamental directed graphs. Acyclic tournament graphs contains any acyclic directed graphs as subgraphs, and undirected bipartite graphs can be regarded as the subgraphs of acyclic tournament graphs by directing each edge from one set of vertices in bipartite graphs to the other. By the elimination theorem(see [4]), reduced Gröbner bases of any subgraphs of acyclic tournament graphs can be obtained automatically if that of acyclic tournament graphs can be calculated. Any elements in reduced Gröbner bases for toric ideals of these graphs correspond to the circuits in the graphs. So we can characterize the reduced Gröbner bases of toric ideals in terms of circuits of graphs. In [13], we have applied the Gröbner bases of acyclic tournament graphs to some hypergeometric systems.

We focus especially on the degree and the number of elements in reduced Gröbner bases. The number of elements in reduced Gröbner bases of graphs are related to the complexity of integer programming problems arising from the graphs.

In this paper, we show that there exist some term orders such that the number of elements in reduced Gröbner bases remain in polynomial order by characterizing the bases in terms of circuits of graphs. We next analyze the number of elements in reduced Gröbner bases with respect to various term orders using TiGERS [9]. We finally discuss the applications to the minimum cost flow problem on acyclic directed graphs.

2 Preliminaries

In this section, we give some basic definitions of Gröbner bases and toric ideals. We refer to [4, 5] for the introductions of Gröbner bases, and [16] for the introductions of toric ideals and their applications.

2.1 Gröbner Bases

Let k be a field and $k[x_1, \ldots, x_n]$ be the polynomial ring in n variables. For a set of variables $\mathbf{x} = (x_1, \ldots, x_n)$ and a non-negative integer vector $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$ ($\mathbb{Z}_{\geq 0}$ means the set of all non-negative integers), we denote $\mathbf{x}^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$.

Definition 2.1 *Let $\succ$ be a total order on $k[x_1, \ldots, x_n]$. We call $\succ$ a* term order *on $k[x_1, \ldots, x_n]$ if it satisfies the following:*

1. $^\forall \alpha, \beta, \gamma \in \mathbb{Z}^n_{\geq 0},\ \mathbf{x}^\alpha \succ \mathbf{x}^\beta \Longrightarrow \mathbf{x}^\alpha \mathbf{x}^\gamma \succ \mathbf{x}^\beta \mathbf{x}^\gamma$.
2. $^\forall \alpha \in \mathbb{Z}^n_{\geq 0} \setminus \{0\},\ \mathbf{x}^\alpha \succ 1$.

For a polynomial f and a term order $\succ$, we call the largest term in f with respect to $\succ$ the initial term *of f and write $in_\succ(f)$, or short, $in(f)$.*

We give some examples of term orders.

Definition 2.2 *Fix a variable ordering $x_{i_1} \succ x_{i_2} \succ \cdots \succ x_{i_n}$. We say $\succ$ is a* purely lexicographic order *induced by this variable ordering if, for any $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$, $\mathbf{x}^\alpha \succ \mathbf{x}^\beta$ if and only if there exists $1 \leq m \leq n$ such that $\alpha_{i_k} = \beta_{i_k}$ for $k < m$ and $\alpha_{i_m} > \beta_{i_m}$.*

Definition 2.3 *Let $\mathbf{c} \in \mathbb{R}^n_{\geq 0}$ be a non-negative vector and $\succ$ an arbitrary term order. We define a* refinement *$\succ_\mathbf{c}$ of $\mathbf{c}$ with respect to $\succ$ as follows: for any $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$,*

$$\mathbf{x}^\alpha \succ_\mathbf{c} \mathbf{x}^\beta \Longleftrightarrow \mathbf{c} \cdot \alpha > \mathbf{c} \cdot \beta \text{ or } (\mathbf{c} \cdot \alpha = \mathbf{c} \cdot \beta \text{ and } \mathbf{x}^\alpha \succ \mathbf{x}^\beta).$$

Proposition 2.4 ([16, Corollary 1.10. and Proposition 1.11.]) *For any term order $\succ$ and any ideal $I \subseteq k[x_1, \ldots, x_n]$, there exists $\mathbf{c} \in \mathbb{R}^n_{\geq 0}$ such that $in_\succ(I) = in_{\succ_\mathbf{c}}(I)$.*

Definition 2.5 *Let $I \subset k[x_1, \ldots, x_n]$ be an ideal and $\succ$ be a term order. A finite subset $\mathcal{G} = \{g_1, \ldots, g_s\} \subseteq I$ is a* Gröbner basis *for I with respect to $\succ$ if the* initial ideal *$in_\succ(I) := \langle in_\succ(f) \colon f \in I \rangle$ is generated by $in_\succ(g_1), \ldots, in_\succ(g_s)$. In addition, Gröbner basis $\mathcal{G}$ is* reduced *if $\mathcal{G}$ satisfies the following:*

1. *For any i, the coefficient of $in_\succ(g_i)$ is 1.*
2. *For any i, any term of g_i is not divisible by $in_\succ(g_j)$ ($i \neq j$).*

Proposition 2.6 *For an ideal and a term order, the reduced Gröbner basis is defined uniquely.*

Proposition 2.7 *For any term order $\succ$, a Gröbner basis for I with respect to $\succ$ is a basis for I.*

Definition 2.8 *Let $I \subseteq k[x_1, \dots, x_n]$ be an ideal. Then the union of all reduced Gröbner bases for I with respect to all term orders is a Gröbner basis for I with respect to any term order. This basis is called the* universal Gröbner basis *for I.*

Although there are infinite term orders, the number of elements in a universal Gröbner basis is finite.

We define "division" on multi-variable polynomial ring.

Theorem 2.9 *Fix a term order $\succ$ and a Gröbner basis $\mathcal{G} = \{g_1, \dots, g_s\}$ for I with respect to $\succ$. Then every $f \in k[x_1, \dots, x_n]$ can be written as*

$$f = a_1 g_1 + \cdots + a_s g_s + r,\ \ a_i, r \in k[x_1, \dots, x_n]$$

where either $r = 0$ or no term of r is divisible by any of $in_\succ(g_1), \dots, in_\succ(g_s)$. Then r is unique, and called a normal form *of f by $\mathcal{G}$.*

2.2 Toric Ideals

Fix a matrix $A \in \mathbb{Z}^{d \times n}$ and let $\mathbf{a}_i$ be the i-th column of A. Each vector $\mathbf{a}_i$ is identified with a monomial $\mathbf{t}^{\mathbf{a}_i}$ in the Laurent polynomial ring $k[\mathbf{t}^{\pm 1}] := k[t_1, \dots, t_d, t_1^{-1}, \dots, t_d^{-1}]$.

Definition 2.10 *Consider the homomorphism*

$$\pi \colon k[x_1, \dots, x_n] \longrightarrow k[\mathbf{t}^{\pm 1}],\ \ x_i \longmapsto \mathbf{t}^{\mathbf{a}_i}.$$

The kernel of π is denoted I_A and called the toric ideal *of A.*

Every vector $\mathbf{u} \in \mathbb{Z}^n$ can be written uniquely as $\mathbf{u} = \mathbf{u}^+ - \mathbf{u}^-$ where $\mathbf{u}^+$ and $\mathbf{u}^-$ are non-negative and have disjoint support.

Lemma 2.11

$$I_A = \langle \mathbf{x}^{\mathbf{u}_i^+} - \mathbf{x}^{\mathbf{u}_i^-} : \mathbf{u}_i \in \ker(A) \cap \mathbb{Z}^n, i = 1, \dots, s \rangle$$

Furthermore, a toric ideal is generated by finite binomials.

Definition 2.12 *A binomial $\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} \in I_A$ is called a* circuit *if the support of $\mathbf{u}$ is minimal with respect to inclusion in $\ker(A)$ and the coordinates of $\mathbf{u}$ are relatively prime. We denote the set of all circuits in I_A by $\mathcal{C}_A$.*

Definition 2.13 *A binomial $\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} \in I_A$ is called* primitive *if there exists no other binomial $\mathbf{x}^{\mathbf{v}^+} - \mathbf{x}^{\mathbf{v}^-} \in I_A$ such that both $\mathbf{u}^+ - \mathbf{v}^+$ and $\mathbf{u}^- - \mathbf{v}^-$ are non-negative. The set of all primitive binomials in I_A is called the* Graver basis *of A and written as Gr_A.*

Let $\mathcal{U}_A$ be the universal Gröbner basis of I_A.

Proposition 2.14 ([16, Proposition 4.11.]) *For any matrix A,*

$$\mathcal{C}_A \subseteq \mathcal{U}_A \subseteq Gr_A.$$

3 Gröbner Bases for Acyclic Tournament Graphs

Let D_n be an acyclic tournament graph with n vertices which have labels $1, 2, \ldots, n$ such that each edge (i, j) $(i < j)$ is directed from i to j. Let $m = \binom{n}{2}$ be the number of edges in D_n. We associate each edge (i, j) with a variable x_{ij} in the polynomial ring $k[\mathbf{x}] := k[x_{ij} : 1 \le i < j \le n]$. Let A_n be the vertex-edge incidence matrix of D_n.

3.1 Toric Ideals of Acyclic Tournament Graphs

A *walk* in D_n is a sequence of vertices $(v_1, v_2, \ldots, v_p)$ such that (v_i, v_{i+1}) or (v_{i+1}, v_i) is an arc of D_n for each $1 \le i < p$. A *cycle* is a walk $(v_1, v_2, \ldots, v_p, v_1)$. A *circuit* is a cycle $(v_1, v_2, \ldots, v_p, v_1)$ such that $v_i \ne v_j$ for any $i \ne j$.

Definition 3.1 *Let C be a circuit of D_n. If we fix a direction of C, we can partition the edges of C into two sets C^+ and C^- such that C^+ is the set of forward edges and C^- is the set of backward edges. Then the vector $\mathbf{u} = (u_{ij})_{1 \le i < j \le n} \in \mathbb{R}^m$ defined by*

$$u_{ij} = \begin{cases} 1 & \text{if } (i,j) \in C^+ \\ -1 & \text{if } (i,j) \in C^- \\ 0 & \text{if } (i,j) \notin C \end{cases}$$

is called the incidence vector *of C.*

Lemma 3.2 ([2, Proposition 2.17]) *A binomial $\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} \in I_{A_n}$ is a circuit if and only if $\mathbf{u}$ is the incidence vector of a circuit of D_n.*

Proposition 3.3 ([16, Exercise 4(8)]) *For the case of I_{A_n}, $\mathcal{C}_{A_n} = \mathcal{U}_{A_n} = Gr_{A_n}$.*

(**Proof**) If $\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} \in Gr_{A_n}$ is not a circuit of I_{A_n}, then there exists a circuit $\mathbf{x}^{\mathbf{c}^+} - \mathbf{x}^{\mathbf{c}^-} \in I_{A_n}$ such that

$$supp(\mathbf{c}^+) \subseteq supp(\mathbf{u}^+), \quad supp(\mathbf{c}^-) \subseteq supp(\mathbf{u}^-).$$

By Lemma 3.2, since each element in $\mathbf{c}^+$ and $\mathbf{c}^-$ is either 0 or 1, $\mathbf{x}^{\mathbf{u}^+}$ is divisible by $\mathbf{x}^{\mathbf{c}^+}$ and $\mathbf{x}^{\mathbf{u}^-}$ is divisible by $\mathbf{x}^{\mathbf{c}^-}$. Then $\mathbf{u}$ is not primitive, which is contradiction. ∎

Corollary 3.4 *The universal Gröbner basis $\mathcal{U}_{A_n}$ is a set of binomials which correspond to all of the circuits of D_n.*

Corollary 3.5 *The number of elements in $\mathcal{U}_{A_n}$ is of exponential order with respect to n.*

An ideal I is called *homogeneous* if for any $f = f_1 + f_2 + \cdots + f_n \in I$ (f_i is the homogeneous component of degree i in f), $f_i \in I$ for any i. Since $x_{12}x_{23} - x_{13} \in I_{A_n}$ and $x_{12}x_{23} \notin I_{A_n}$, I_{A_n} is not homogeneous for the standard positive grading $\deg(x_{ij}) = 1$ $({}^\forall i, j)$.

Corollary 3.6 *I_{A_n} is not homogeneous for the grading $\deg(x_{ij}) = 1$ $({}^\forall i, j)$.*

But we can change the positive grading such that I_{A_n} becomes homogeneous.

Theorem 3.7 *If we set a positive grading as*

$$\deg(x_{ij}) = j - i, \quad 1 \le i < j \le n, \tag{1}$$

then I_{A_n} is a homogeneous ideal.

Lemma 3.8 (**[4, Chapter 8, §3. Theorem 2.]**) *An ideal I is homogeneous if and only if $I = \langle f_1, \ldots, f_s \rangle$ where $f_1, \ldots, f_s$ are homogeneous polynomials.*

(**Proof of Theorem 3.7**) By the above lemma, it suffices to show that any element in the universal Gröbner basis $\mathcal{U}_{A_n}$ is a homogeneous polynomial with respect to the positive grading (1).

Let $C = (v_1, v_2, \ldots, v_p, v_1)$ be a circuit in D_n. Let $C^+ := \{k\colon v_k < v_{k+1}\}$ and $C^- := \{k\colon v_k > v_{k+1}\}$ (we set $v_{p+1} := v_1$). The binomial f_C corresponding to C is

$$f_C = \prod_{k \in C^+} x_{v_k v_{k+1}} - \prod_{k \in C^-} x_{v_{k+1} v_k}.$$

Then, since $C^+ \cap C^- = \emptyset$,

$$\begin{aligned} \deg\left(\prod_{k \in C^+} x_{v_k v_{k+1}}\right) - \deg\left(\prod_{k \in C^-} x_{v_{k+1} v_k}\right) &= \sum_{k \in C^+} (v_{k+1} - v_k) - \sum_{k \in C^-} (v_k - v_{k+1}) \\ &= \sum_{k=1}^{p} (v_{k+1} - v_k) \\ &= 0 \end{aligned}$$

Thus f_C is homogeneous. ∎

3.2 Some Gröbner bases of I_{A_n}

In this section, we show that the elements in reduced Gröbner bases with respect to some specific term orders can be given in terms of graphs. As a corollary, we can show that there exist term orders for which reduced Gröbner bases remain in polynomial order.

Remark 3.9 *In this section, we line under the initial term of each polynomial.*

Theorem 3.10 *Let $\succ$ be the purely lexicographic order induced by the following variable ordering:*

$$x_{ij} \succ x_{kl} \Longleftrightarrow i < k \text{ or } (i = k \text{ and } j < l).$$

Then the reduced Gröbner basis for I_{A_n} with respect to $\succ$ is

$$\{g_{ijk}\colon 1 \le i < j < k \le n\} \cup \{g_{ijkl}\colon 1 \le i < j < k < l \le n\} \tag{2}$$

where

$$\begin{aligned} g_{ijk} &:= \underline{x_{ij} x_{jk}} - x_{ik} \ (1 \le i < j < k \le n) \\ g_{ijkl} &:= \underline{x_{ik} x_{jl}} - x_{il} x_{jk} \ (1 \le i < j < k < l \le n). \end{aligned}$$

In particular, the number of elements in this Gröbner basis equals $\binom{n}{3} + \binom{n}{4}$.

The set $\{g_{ijk}\colon 1 \le i < j < k \le n\}$ corresponds to all of the circuits of length three in D_n, and $\{g_{ijkl}\colon 1 \le i < j < k < l\}$ corresponds to some of the circuits of length four(Figure 1).

(**Proof**) It suffices to show that any binomial which corresponds to the circuit in D_n is either a element of Gröbner basis or divisible by the initial term of some element of Gröbner basis (2).

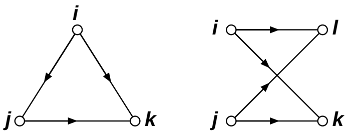

Figure 1: The circuit corresponding to g_{ijk} (left) and the circuit corresponding to g_{ijkl} (right).

For any circuit of length three defined by three vertices $i < j < k$, the associated binomial equals $\underline{x_{ij}x_{jk}} - x_{ik}$, which is g_{ijk}.

The circuits defined by four vertices $i < j < k < l$ are $C_1 := (i, j, k, l, i)$, $C_2 := (i, j, l, k, i)$, $C_3 := (i, k, j, l, i)$ and their opposites. The binomial which corresponds to C_1 or its opposite is $\underline{x_{ij}x_{jk}x_{kl}} - x_{il}$, whose initial term is divisible by $in_{\succ}(g_{ijk})$. Similarly, the initial term of binomial which corresponds to C_2 or its opposite is divisible by $in_{\succ}(g_{ijl})$. The binomial which corresponds to C_3 or its opposite is g_{ijkl}.

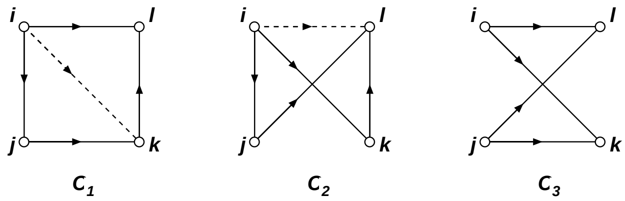

Figure 2: The circuits C_1, C_2, C_3.

Let C be a circuit of length more than five. Let v_1 be the vertex whose label is minimum in C, and $C := (v_1, v_2, \ldots, v_p, v_1)$. Without loss of generality, we set $v_2 < v_p$. Let f_C be the binomial corresponding to C, then $in_{\succ}(f_C)$ is product of all variables whose associated edges have the same direction as (v_1, v_2) on C. We show that $in_{\succ}(f_C)$ is divisible by initial term of some g_{ijk} or g_{ijkl}, which implies that (2) is Gröbner basis for I_{A_n} with respect to $\succ$.

If $v_2 < v_3$, then (v_1, v_2) and (v_2, v_3) have the same direction on C. Thus both $x_{v_1v_2}$ and $x_{v_2v_3}$ appear in $in_{\succ}(f_C)$, and $in_{\succ}(f_C)$ is divisible by $in_{\succ}(g_{v_1v_2v_3})$ (Figure 3 left).

If $v_2 > v_3$, then since $v_3 < v_2 < v_p$, there exists k $(3 \leq k \leq p-1)$ such that $v_1 < v_k < v_2 < v_{k+1}$. Then both $x_{v_1v_2}$ and $x_{v_kv_{k+1}}$ appear in $in_{\succ}(f_C)$, and $in_{\succ}(f_C)$ is divisible by $in_{\succ}(g_{v_1v_kv_2v_{k+1}})$ (Figure 3 right).

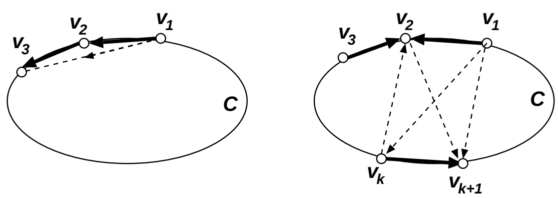

Figure 3: $x_{v_1v_2}$ and $x_{v_2v_3}$ (left) or $x_{v_1v_2}$ and $x_{v_kv_{k+1}}$ (right) appear in $in_{\succ}(f_C)$.

Any term of g_{ijk} is not divisible by the initial term of any other binomials g_{ijk} or g_{ijkl}, and so as g_{ijkl}. This implies that (2) is reduced. ∎

Corollary 3.11 *Let $\succ$ be any term order and $\mathbf{c} = (c_{12}, \ldots, c_{1n}, c_{23}, \ldots, c_{n-1,n}) \in \mathbb{R}^m_{\geq 0}$ satisfies* *the following two conditions:*

- $c_{ij} + c_{jk} > c_{ik}$ *for any* $1 \le i < j < k \le n$ *and*
- $c_{ik} + c_{jl} > c_{il} + c_{jk}$ *for any* $1 \le i < j < k < l \le n$.

Then the reduced Gröbner basis for I_{A_n} *with respect to* $\succ_{\mathbf{c}}$ *equals the basis* (2) *in Theorem 3.10.*

(**Proof**) Let $\succ'$ be the purely lexicographic order defined in Theorem 3.10. Then $in_{\succ_{\mathbf{c}}}(g_{ijk}) = x_{ij}x_{jk} = in_{\succ'}(g_{ijk})$ since $c_{ij} + c_{jk} > c_{ik}$, and $in_{\succ_{\mathbf{c}}}(g_{ijkl}) = x_{ik}x_{jl} = in_{\succ'}(g_{ijkl})$ since $c_{ik} + c_{jl} > c_{il} + c_{jk}$. Thus $in_{\succ_{\mathbf{c}}}(I_{A_n}) = in_{\succ'}(I_{A_n})$, which implies that the reduced Gröbner basis for I_{A_n} with respect to $\succ_{\mathbf{c}}$ equals the basis (2). ∎

Theorem 3.12 *Let* $\succ$ *be the purely lexicographic order induced by the following variable ordering:*

$$x_{ij} \succ x_{kl} \Longleftrightarrow j - i < l - k \text{ or } (j - i = l - k \text{ and } i < k).$$

Then the reduced Gröbner basis for I_{A_n} *with respect to* $\succ$ *is*

$$\{g_{ijk} \colon 1 \le i < j < k \le n\} \cup \{g_{ijkl} \colon 1 \le i < j < k < l \le n\} \tag{3}$$

where

$$\begin{aligned} g_{ijk} &:= \underline{x_{ij}x_{jk}} - x_{ik} \ (1 \le i < j < k \le n) \\ g_{ijkl} &:= \underline{x_{il}x_{jk}} - x_{ik}x_{jl} \ (1 \le i < j < k < l \le n). \end{aligned}$$

In particular, the number of elements in this Gröbner basis equals $\binom{n}{3} + \binom{n}{4}$.

The set $\{g_{ijk} \colon 1 \le i < j < k \le n\}$ corresponds to all of the circuits of length three in D_n, and $\{g_{ijkl} \colon 1 \le i < j < k < l\}$ corresponds to the set of circuits of length four same as in Figure 1, but the direction of each circuit is opposite.

(**Proof**) For any circuit of length three defined by three vertices $i < j < k$, the associated binomial equals $\underline{x_{ij}x_{jk}} - x_{ik}$, which is g_{ijk}.

The circuits defined by four vertices $i < j < k < l$ are $C_1 := (i, j, k, l, i)$, $C_2 := (i, j, l, k, i)$, $C_3 := (i, k, j, l, i)$ and their opposites. The binomial which corresponds to C_1 or its opposite is $\underline{x_{ij}x_{jk}x_{kl}} - x_{il}$, whose initial term is divisible by $in_{\succ}(g_{ijk})$. The binomial which corresponds to C_2 or its opposite is $x_{ij}x_{jl} - x_{ik}x_{kl}$. If its initial term is $x_{ij}x_{jl}$, it is divisible by $in_{\succ}(g_{ijl})$. If initial term is $x_{ik}x_{kl}$, it is divisible by $in_{\succ}(g_{ikl})$. The binomial which corresponds to C_3 or its opposite is g_{ijkl}.

Let C be a circuit of length more than five. Let (v_1, v_2) $(v_1 < v_2)$ be the edge which the difference of labels is minimum in C, and $C := (v_1, v_2, \ldots, v_p, v_1)$. Let f_C be the binomial corresponding to C, then $in_{\succ}(f_C)$ is product of all variables whose associated edges have the same direction with (v_1, v_2) on C.

If $v_2 < v_3$, then both $x_{v_1v_2}$ and $x_{v_2v_3}$ appear in $in_{\succ}(f_C)$, and $in_{\succ}(f_C)$ is divisible by $in_{\succ}(g_{v_1v_2v_3})$. Similarly, if $v_p < v_1$, then $in_{\succ}(f_C)$ is divisible by $in_{\succ}(g_{v_pv_1v_2})$.

Let $v_3 < v_2$ and $v_1 < v_p$. Then $v_3 < v_1 < v_2 < v_p$ by the definition of v_1 and v_2. If there exists some q such that $v_q < v_{q+1} < v_{q+2}$, then $in_{\succ}(f_C)$ is divisible by $in_{\succ}(g_{v_qv_{q+1}v_{q+2}})$. We show that when there does not exist such q, there exists some r $(3 \le r \le p-1)$ such that $v_r < v_1 < v_2 < v_{r+1}$ (Figure 4 left)).

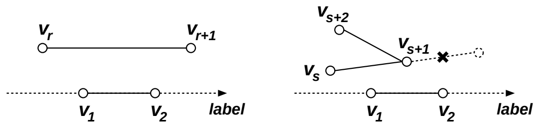

Figure 4: $v_r < v_1 < v_2 < v_{r+1}$ ($^\exists r$) (left). If $v_s < v_1 < v_{s+1} < v_2$, it must be $v_{s+2} < v_1$ (right).

Let $v_s < v_1 < v_{s+1} < v_2$ (Figure 4 right). Then $v_{s+2} < v_{s+1}$, and $v_{s+2} < v_1$ by the definition of v_1 and v_2. Thus there must be some r ($3 \le r \le p-1$) such that $v_r < v_1 < v_2 < v_{r+1}$ since $v_3 < v_1 < v_2 < i_p$.

Then $in_\succ(f_C)$ is divisible by $in_\succ(g_{v_r v_1 v_2 v_{r+1}})$.

Any term of g_{ijk} is not divisible by the initial term of any other binomials g_{ijk} or g_{ijkl}, and so as g_{ijkl}. This implies that (3) is reduced. ∎

Corollary 3.13 *Let $\succ$ be any term order and $\mathbf{c} = (c_{12}, \dots, c_{1n}, c_{23}, \dots, c_{n-1,n}) \in \mathbb{R}^m_{\ge 0}$ satisfies the following two conditions:*

- $c_{ij} + c_{jk} > c_{ik}$ *for any* $1 \le i < j < k \le n$ *and*
- $c_{il} + c_{jk} > c_{ik} + c_{jl}$ *for any* $1 \le i < j < k < l \le n$.

Then the reduced Gröbner basis for I_{A_n} with respect to $\succ_\mathbf{c}$ equals the basis (3) *in Theorem 3.12.*

(**Proof**) Let $\succ'$ be the purely lexicographic order defined in Theorem 3.12. Then $in_{\succ_\mathbf{c}}(g_{ijk}) = x_{ij}x_{jk} = in_{\succ'}(g_{ijk})$ since $c_{ij} + c_{jk} > c_{ik}$, and $in_{\succ_\mathbf{c}}(g_{ijkl}) = x_{il}x_{jk} = in_{\succ'}(g_{ijkl})$ since $c_{il} + c_{jk} > c_{ik} + c_{jl}$. Thus $in_{\succ_\mathbf{c}}(I_{A_n}) = in_{\succ'}(I_{A_n})$, which implies that the reduced Gröbner basis for I_{A_n} with respect to $\succ_\mathbf{c}$ equals the basis (3). ∎

Theorem 3.14 *Let $\succ$ be the purely lexicographic order induced by the following variable ordering:*

$$x_{ij} \succ x_{kl} \Longleftrightarrow i < k \text{ or } (i = k \text{ and } j > l).$$

Then the reduced Gröbner basis for I_{A_n} with respect to $\succ$ is

$$\{g_{ij} \colon 1 \le i < j-1 < n\} \tag{4}$$

where

$$g_{ij} := \underline{x_{ij}} - x_{i,i+1}x_{i+1,i+2}\cdots x_{j-1,j} \ (1 \le i < j-1 < n).$$

In particular, the number of elements in this Gröbner basis equals $\binom{n}{2} - (n-1)$.

The set $\{g_{ij} \colon 1 \le i < j-1 < n\}$ corresponds to all of the fundamental circuits of D_n for the spanning tree $T := \{(i, i+1) \colon 1 \le i < n\}$.

(**Proof**) Let C be a circuit which is not the fundamental circuit of T. Let v_1 be the vertex whose label is minimum in C, and $C := (v_1, v_2, \dots, v_p, v_1)$. Without loss of generality, we set $v_2 < v_p$. Then the variable $x_{v_1 v_p}$ appears in the initial term of associated binomial f_C. Thus $in_\succ(f_C)$ is divisible by $in_\succ(g_{v_1 v_p})$.

The initial term of g_{ij} corresponds to an edge (i,j) which is not contained in T, and other term corresponds to several edges which are contained in T. Thus any term of g_{ij} is not divisible by the initial term of other binomial in (4), which implies that (4) is reduced. ∎

Corollary 3.15 *Let $\succ$ be any term order and* $\mathbf{c} = (c_{12}, \dots, c_{1n}, c_{23}, \dots, c_{n-1,n}) \in \mathbb{R}^m_{\geq 0}$ *satisfies the following condition:*

$$c_{ij} > c_{i,i+1} + c_{i+1,i+2} + \cdots + c_{j-1,j} \text{ for any } 1 \leq i < j-1 < n.$$

Then the reduced Gröbner basis for I_{A_n} with respect to $\succ_{\mathbf{c}}$ equals the basis (4) *in Theorem 3.14.*

(**Proof**) Let $\succ'$ be the purely lexicographic order defined in Theorem 3.14. Then $in_{\succ_{\mathbf{c}}}(g_{ij}) = x_{ij} = in_{\succ'}(g_{ij})$ since $c_{ij} > c_{i,i+1} + c_{i+1,i+2} + \cdots + c_{j-1,j}$. Thus $in_{\succ_{\mathbf{c}}}(I_{A_n}) = in_{\succ'}(I_{A_n})$, which implies that the reduced Gröbner basis for I_{A_n} with respect to $\succ_{\mathbf{c}}$ equals the basis (4). ∎

3.3 Gröbner Bases for Acyclic Directed Graphs

Gröbner bases for acyclic directed graphs and (undirected) bipartite graphs can be obtained from those for acyclic tournament graphs automatically.

Let B_n be the vertex-edge incidence matrix of acyclic directed graph G_n with n vertices and $C_{m,n}$ that of bipartite graph $K_{m,n}$ with vertex sets V, W such that $|V| = m$, $|W| = n$.

We consider G_n as a subgraph of D_n, and let

$$E' := \{(i,j) \colon (i,j) \in E(D_n) \setminus E(G_n)\}$$

where $E(D_n)$ (resp. $E(G_n)$) is the edge set of D_n (resp. G_n).

Proposition 3.16 $I_{B_n} = I_{A_n} \cap k[x_{ij} \colon (i,j) \notin E']$.

(**Proof**) If $f = \mathbf{x}^{\mathbf{c}^+} - \mathbf{x}^{\mathbf{c}^-} \in I_{B_n}$, there exists a cycle C in G_n such that for a suitable orientation of C, the support of $\mathbf{c}^+$ is the set of forward edges in C and the support of $\mathbf{c}^-$ is the set of backward edges in C. Then C is also a cycle in D_n, which implies that $f \in I_{A_n} \cap k[x_{ij} \colon (i,j) \notin E']$.

Conversely, Let $f = \mathbf{x}^{\mathbf{c}^+} - \mathbf{x}^{\mathbf{c}^-} \in I_{A_n} \cap k[x_{ij} \colon (i,j) \notin E']$. Since $f \in I_{A_n}$, there exists a cycle C in D_n such that for a suitable orientation of C, the support of $\mathbf{c}^+$ is the set of forward edges in C and the support of $\mathbf{c}^-$ is the set of backward edges in C. Furthermore, since $f \in k[x_{ij} \colon (i,j) \notin E']$, C contains no edge which is contained in E'. Then C is also a cycle in G_n, which implies that $f \in I_{B_n}$. ∎

Let $G_{m,n}$ be a subgraph of D_{m+n} such that the edge set $E(G_{m,n})$ of $G_{m,n}$ is

$$E(G_{m,n}) := \{(i,j) \colon 1 \leq i \leq m \text{ and } m+1 \leq j \leq m+n\}.$$

$G_{m,n}$ is obtained from $K_{m,n}$ by orienting each edge from the vertex in V to the vertex in W. Let $C'_{m,n}$ be the vertex-edge incidence matrix of $G_{m,n}$.

Proposition 3.17 $I_{C'_{m,n}} = I_{C_{m,n}} = I_{D_{m+n}} \cap k[x_{ij} \colon (i,j) \in E(G_{m,n})]$.

(**Proof**) The i-th row of $C'_{m,n}$ equals the i-th row of $C_{m,n}$ for $1 \leq i \leq m$ and (-1) times the i-th row of $C_{m,n}$ for $m+1 \leq i \leq m+n$. Thus $I_{C'_{m,n}} = I_{C_{m,n}}$ since $\ker(C'_{m,n}) = \ker(C_{m,n})$.

The proof of the second equality is similar to that of Proposition 3.16. ∎

By Proposition 3.16 and Proposition 3.17, Gröbner bases of acyclic directed graphs and bipartite graphs can be obtained from those for acyclic tournament graphs automatically using the following *elimination theorem.*

Theorem 3.18 ([4, Chapter 3, §3. Theorem 2 and Exercise 5]) *Fix an integer $1 \leq l \leq n$ and let $\succ$ be a term order on $k[x_1, x_2, \dots, x_n]$ such that any monomial involving at least one of $x_1, \dots, x_l$ is greater than all monomials in $k[x_{l+1}, \dots, x_n]$. Let $\succ'$ be a term order which is the restriction of $\succ$ to $k[x_{l+1}, \dots, x_n]$. If I is an ideal in $k[x_1, x_2, \dots, x_n]$ and $\mathcal{G}$ is a Gröbner basis of I with respect to $\succ$, then $\mathcal{G} \cap k[x_{l+1}, \dots, x_n]$ is a Gröbner basis for $I \cap k[x_{l+1}, \dots, x_n]$ with respect to $\succ'$.*

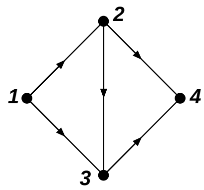

Figure 5: G_4: subgraph of D_4

Example 3.19 *Let $n = 4$ and consider the following subgraph G_4 of D_4. Let $\succ'$ be the lexicographic order on $k[x_{12}, x_{13}, x_{23}, x_{24}, x_{34}]$ induced by the variable ordering:*

$$x_{13} \succ x_{12} \succ x_{24} \succ x_{23} \succ x_{34}.$$

When we want to calculate the Gröbner basis $\mathcal{G}'$ of G_4 with respect to $\succ'$, we first fix the term order $\succ$ on $k[x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}]$. Let $\succ$ be the lexicographic order induced by the variable ordering

$$x_{14} \succ x_{13} \succ x_{12} \succ x_{24} \succ x_{23} \succ x_{34}.$$

Then the Gröbner basis $\mathcal{G}$ of D_4 is

$$\mathcal{G} = \{x_{13} - x_{12}x_{23},\ \ x_{14} - x_{12}x_{23}x_{34},\ \ x_{24} - x_{23}x_{34}\}$$

by Theorem 3.14. Thus using the elimination theorem,

$$\begin{aligned} \mathcal{G}' &= \mathcal{G} \cap k[x_{12}, x_{13}, x_{23}, x_{24}, x_{34}] \\ &= \{x_{13} - x_{12}x_{23},\ \ x_{24} - x_{23}x_{34}\}. \end{aligned}$$

∎

4 Bounds for Size of Gröbner Bases for Various Term Orders

In this section, we deal with the degree and the number of elements in reduced Gröbner bases with respect to various term orders. Generally the degree of reduced Gröbner bases for toric ideals is at most of exponential order [15], but the number of elements is not well understood. For the case of toric ideals of acyclic tournament graphs, since those vertex-edge incidence matrices are unimodular, the degree and the size of reduced Gröbner bases may be bounded.

As we have shown in Corollary 3.6 and Theorem 3.7, I_{A_n} is not homogeneous for the positive grading $\deg(x_{ij}) = 1$ but homogeneous for the positive grading $\deg(x_{ij}) = j - i$. We call the former *standard grading* and the latter *graphical grading.*

4.1 Bound for Degree of Gröbner Bases

Since the elements of toric ideals of acyclic tournament graphs correspond to the circuits in the graphs, we can bound the degree of elements in reduced Gröbner bases in both of the cases graphic grading and standard grading.

4.1.1 Case of Graphical Grading

We first consider the case of graphical grading.

Theorem 4.1 *The lower bound for the degree of elements in reduced Gröbner bases for I_{A_n} is $n-2$.*

(**Proof**) It suffices to show that any reduced Gröbner bases contain the binomial of degree more than $n-2$.

Because of the definition of Gröbner basis, any reduced Gröbner basis has an element g such that $in(g)$ divides the initial term of the binomial $f := x_{1,n-1}x_{n-1,n} - x_{1n}$ corresponding the cycle $(1, n-1, n, 1)$.

If $in(f) = x_{1n}$, then $in(g) = x_{1n}$ and the degree of $in(g)$ equals $n-1$. If $in(f) = x_{1,n-1}x_{n-1,n}$, then $in(g)$ must contain the variable $x_{1,n-1}$. In fact, if $in(g)$ does not contain $x_{1,n-1}$, then $in(g) = x_{n-1,n}$. But any cycle which passes the edge $(n-1, n)$ always passes at least one of the edge $(i, n-1)$ $(1 \leq i \leq n-2)$ from the vertex i to the vertex $n-1$, $in(g)$ contains the variable $x_{i,n-1}$, this is contradiction. Thus $\deg(g) \geq n-2$. ∎

Theorem 4.2 *The upper bound for the degree of elements in reduced Gröbner bases for I_{A_n} is $O(n^2)$.*

(**Proof**) The length of each circuit in D_n is at most n. But the direction of at least one edge is opposite since D_n is acyclic. Thus each term of elements in reduced Gröbner bases contains at most $n-1$ variables. Since the degree of each variable is less than $n-1$, the degree of each element in reduced Gröbner bases is at most $(n-1)^2 = O(n^2)$. ∎

4.1.2 Case of Standard Grading

We next consider the case of standard grading. In this case, the degree becomes linear order.

Theorem 4.3 *The minimum value of the degree of elements in reduced Gröbner bases for I_{A_n} is 2. The basis we have shown in Theorem 3.10 is the example achieving this bound.*

(**Proof**) The length of the circuit in D_n is at least 3, but the direction of at least one edge is opposite. Thus the degree of any elements in reduced Gröbner bases is at least 2. ∎

Theorem 4.4 *The maximum value of the degree of elements in reduced Gröbner bases for I_{A_n} is $n-1$. The basis we have shown in Theorem 3.14 is the example achieving this bound.*

(**Proof**) The length of the circuit in D_n is at most n. But the direction of at least one edge is opposite since D_n is acyclic. Thus the number of edges in circuit whose direction are same is at most $n-1$, which implies the upper bound of the degree is $n-1$. ∎

4.2 Bound for Number of Elements in Gröbner Bases

For the number of elements in reduced Gröbner bases, we can get lower bound by Proposition 2.7.

Theorem 4.5 *The minimum number of elements in reduced Gröbner bases for I_{A_n} is $\binom{n}{2} - (n-1)$. The basis we have shown in Theorem 3.14 is the example achieving this bound.*

(**Proof**) Because of Proposition 2.7, the number of elements in reduced Gröbner basis is more than the number of elements in the basis for I_{A_n}. Since I_{A_n} corresponds to the cycle space of D_n, the number of elements in the basis for I_{A_n} equals the dimension of the cycle space, which is $\binom{n}{2} - (n-1)$. ∎

To analyze the upper bound for the number of elements in reduced Gröbner bases, we calculate all reduced Gröbner bases for small n using TiGERS [9]. TiGERS is a software system implemented in C which computes the state polytope of a homogeneous toric ideal [10]. Table 1 is the result for $n = 4, 5, 6, 7$.

n	# variables	# GB	max. of elements	min. of elements
4	6	10	5	3
5	10	211	15	6
6	15	48312	37	10
7	21	$\geq$ 37665	$\geq$ 75	15

Table 1: The number of reduced Gröbner basis, maximum of the number of elements and minimum of the number of elements.

For $n \leq 5$, the reduced Gröbner basis in Theorem 3.10 is the example achieving maximum elements, but it is not for $n \geq 6$. For $n = 6$, the Gröbner bases of size 37 are not the bases with respect to purely lexicographic orders. Thus the reduced Gröbner bases which achieve the maximum number of elements seem to be complicated and difficult to characterize.

Question 4.6 *Are the number of elements in reduced Gröbner bases of I_{A_n} of polynomial order with respect to n?*

5 Application to Integer Programming

In this section, we apply the toric ideals I_{A_n} to the minimum cost flow problem.

5.1 Conti-Traverso Algorithm

Conti and Traverso [3] introduced an algorithm based on Gröbner basis to solve integer programs. We describe the condensed version of Conti-Traverso Algorithm(See [16]). This version is useful for highlighting the main computational step involved. For the original version of Conti-Traverso Algorithm, see [3].

Let $A \in \mathbb{Z}^{d\times n}$, $\mathbf{b} \in \mathbb{Z}^d$, $\mathbf{c} \in \mathbb{R}^n_{\geq 0}$. We consider the integer program

$$IP_{A,\mathbf{c}}(\mathbf{b}) := minimize\{\mathbf{c}\cdot\mathbf{x} \colon A\mathbf{x} = \mathbf{b}, \mathbf{x} \in \mathbb{Z}^n_{\geq 0}\}.$$

Conti-Traverso Algorithm is a algorithm using the toric ideal I_A which calculates $\mathbf{x}$ such that $\mathbf{x}$ is minimum in the set $\{A\mathbf{x} = \mathbf{b}, \mathbf{x} \in \mathbb{Z}^n_{\geq 0}\}$ with respect to the term order $\succ_{\mathbf{c}}$, that is one of the optimal solutions of $IP_{A,\mathbf{c}}$. Let $IP_{A,\succ_{\mathbf{c}}}(\mathbf{b})$ the problem that calculate this $\mathbf{x}$.

Algorithm 5.1 (Conti-Traverso Algorithm)
Input: $A \in \mathbb{Z}^{d\times n}$, $\mathbf{b} \in \mathbb{Z}^d$, $\mathbf{c} \in \mathbb{R}^n_{\geq 0}$
Output: *An optimal solution* $\mathbf{u}'$ *for* $IP_{A,\succ_{\mathbf{c}}}(\mathbf{b})$
1. *Compute the reduced Gröbner basis* $\mathcal{G}_{\succ_{\mathbf{c}}}$ *of* I_A *with respect to* $\succ_{\mathbf{c}}$.
2. *For any solution* $\mathbf{u}$ *of* $IP_{A,\mathbf{c}}(\mathbf{b})$*, compute the normal form* $\mathbf{x}^{\mathbf{u}'}$ *of* $\mathbf{x}^{\mathbf{u}}$ *by* $\mathcal{G}_{\succ_{\mathbf{c}}}$.
3. *Output* $\mathbf{u}'$. $\mathbf{u}'$ *is the unique optimal solution of* $IP_{A,\succ_{\mathbf{c}}}(\mathbf{b})$.

Conti-Traverso algorithm has given insight into the structure of integer programming by associating reduced Gröbner bases with *test sets* in integer programming [17].

5.2 Application to Minimum Cost Flow Problem

Using Algorithm 5.1, reduced Gröbner bases for I_{A_n} can be applied to the minimum cost flow problems on D_n or the subgraphs of D_n, or to the transportation problem on the bipartite graphs $K_{m,n}$.

The minimum cost flow problem can be solved by the *cycle canceling algorithm*, that is, for a feasible flow the algorithm iteratively finds a negative cost directed cycle in the residual network and augments flows on this cycle. If the residual network contains no negative cost cycle, then the flow is the minimum cost flow [1]. The *minimum mean cycle-canceling algorithm* [8] is known as a strongly polynomial time algorithm which depends only on the number of vertices and edges. Using this algorithm, from any feasible flow, we can obtain the minimum cost flow by canceling minimum mean cycle at most $O(nm^2 \log n)$ times where n (resp. m) is the number of vertices (resp. edges).

Conti-Traverso algorithm for the minimum cost flow shows that we can obtain the minimum cost flow by augmenting flows only on the negative cost directed cycles which correspond to the reduced Gröbner bases. Thus it is necessary for the study of efficiency of Conti-Traverso algorithm to analyze the size of reduced Gröbner bases and the number of cycle-canceling for the case of acyclic tournament graphs. Although the result in Section 4 shows that it is difficult to analyze the size of reduced Gröbner bases, Conti-Traverso algorithm is efficient since the number of cycles to augment (i.e. the number of elements in reduced Gröbner basis) is much smaller than that in general cycle canceling algorithms (i.e. the number of all cycles in D_n).

Question 5.2 *Can the time complexity of Conti-Traverso algorithm for minimum cost flow be bounded with respect to* n*?*

In [11], we have studied the number of cycle-canceling when we apply Conti-Traverso algorithm for the case of acyclic tournament graphs, acyclic directed graphs and bipartite graphs.

6 Conclusions

In this paper, we have studied the reduced Gröbner bases for toric ideals of acyclic tournament graphs and applied them to the minimum cost flow problems.

The universal Gröbner basis of acyclic tournament graph is of exponential size. We have shown three reduced Gröbner bases whose size is of polynomial order. And we showed the experimental result for the size of reduced Gröbner bases. But the upper bound for the number of elements is not known. Analyzing the upper bound for the number of elements should be a future work.

We also showed an application to the minimum cost flow problems. We can apply the reduced Gröbner bases of acyclic tournament graphs to the minimum cost flow problems using Conti-Traverso Algorithm. This algorithm is similar to the minimum mean cycle-canceling algorithm. But the complexity of cycle-canceling is not known. Analyzing the complexity of this algorithm should be also a future work.

Acknowledgement

The authors thank Mr. Fumihiko Takeuchi for useful comments.

References

[1] R. K. Ahuja, T. L. Magnanti and J. B. Orlin. *Network Flows: Theory, Algorithms, and Applications.* Prentice Hall, New Jersey, 1993.

[2] A. Bachem and W. Kern. *Linear Programming Duality.* Springer-Verlag, Berlin, 1991.

[3] P. Conti and C. Traverso. Buchberger algorithm and integer programming. In *Proc. AAECC-9*, Springer, LNCS **539**(1991), pp. 130–139.

[4] D. A. Cox, J. B. Little and D. B. O'Shea. *Ideals, Varieties, and Algorithms.* Second Edition, Springer-Verlag, New York, 1996.

[5] D. A. Cox, J. B. Little and D. B. O'Shea. *Using Algebraic Geometry.* Springer-Verlag, New York, 1998.

[6] J. A. de Loera, B. Sturmfels and R. R. Thomas. Gröbner bases and triangulations of the second hypersimplex. *Combinatorica*, **15**(1995), pp. 409–424.

[7] P. Diaconis and B. Sturmfels. Algebraic algorithms for sampling from conditional distributions. *Annals of Statistics*, **26**(1998), pp. 363–397.

[8] A. V. Goldberg and R. E. Tarjan. Finding minimum-cost circulations by cancelling negative cycles. *J. ACM.* **36**(1989), pp. 873–886.

[9] B. Huber and R. R. Thomas. TiGERS. `http://www.math.tamu.edu/~rekha/TiGERS_0.9.uu`

[10] B. Huber and R. R. Thomas. Computing Gröbner fans of toric ideals. to appear in *Experimental Mathematics.*

[11] T. Ishizeki. Gröbner Bases of Acyclic Directed Graphs and Reductions in Conti-Traverso Algorithm. *RIMS Kokyuroku: Algebraic Combinatorics on Convex Polytopes*, to appear.

[12] T. Ishizeki and H. Imai. Complexity of Gröbner bases for toric ideals of acyclic tournament graphs. *RIMS Kokyuroku: Foundations of Computer Science*, **1148**(2000), pp. 134–139.

[13] T. Ishizeki and H. Imai. Gröbner bases of acyclic tournament graphs and hypergeometric systems on the group of unipotent matrices. to appear in *RIMS Kokyuroku: Algorithms for D-modules.*

[14] H. Ohsugi and T. Hibi. Toric Ideals Generated by Quadratic Binomials. *J. Algebra*, **218**(1999), pp. 509–527.

[15] B. Sturmfels. Gröbner Bases of Toric Varieties. *Tôhoku Mathematical Journal*, **43**(1991), pp. 249–261.

[16] B. Sturmfels. *Gröbner Bases and Convex Polytopes.* AMS University Lecture Series,**8**, Providence, RI, 1995.

[17] B. Sturmfels and R. R. Thomas. Variation of Cost Functions in Integer Programming. *Mathematical Programming*, **77**(1997), pp. 357–387.