最小費用流問題の双対問題におけるトーリックイデアルの解析

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要旨

近年, 整数線形計画問題に対して代数的な視点からトーリックイデアルを用いたアプローチが行われており, 具体的にはそのグレブナ基底や standard pair を用いた手法が注目されている. 本研究では, 無閉路トーナメントグラフ上の最小費用流問題を対象とし, その双対問題を考察する. まずトーリックイデアルの全てのグレブナ基底はサーキットをなすことを示す. 次いで特にコストベクトルが負であるときを取り上げ, この時グレブナ基底のサイズが最小であること, またグレブナ基底より得られた standard pair より d を点の数として arithmetic degree が $(d-1)!$ となること, および standard pair と主問題における実行可能な全域木が対応することを示す. また, 双対問題の arithmetic degree は主問題と異なり, 最小の場合でも指数オーダーになるとの予想を提示する.

Analysis of toric ideal on dual problem of minimum cost flow

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Abstract

Recently algebraical approaches using toric ideal have been carried out, and now the methods with Gröbner basis and standard pair are paid attention to. In this study, we focus to minimum cost flow problem on an acyclic tournament graph and investigate its dual problem. First we prove that universal Gröbner basis of toric ideal is associated with circuit. Next we focus the case that a cost vector is negative, we explain that the size of Gröbner basis becomes minimum, that by standard pairs generated by Gröbner basis arithmetic degree is $(d-1)!$ where d is the number of vertices, and that one standard pair corresponds to one feasible spanning tree in the graph in primal problem. Then we suggest a conjecture that arithmetic degree in dual problems is of exponential order even in minimum cases, as opposed to primal problems.

1 Introduction

The methods to solve linear problems have been researched for many years. Especially, Karmarker's algorithm [9] is famous as a method to solve such problems in polynomial time. And integer programming problems, in which variables are all integer, are famous as representative problems of linear programming problem. This problem is known as NP-hard, thus much investigation about approxmiate method has been done.

But recently, some algebraic approaches come to be applied to integer programming problem. In these approaches, Gröbner bases and standard pairs are useful tools (See [1] and [3]). Although they do not give improvement of complexity in comparison with existing methods, these methods are interesting in terms of algebraic view.

The minimum cost flow problem, which is a restricted case of integer programming problem, is well-known as the problem which can be solved in polynomial time. Gröbner basis approach is based on cycle-canceling algorithm [13, 15], meanwhile standard pair approach is that by solving equations for each gained standard pair [3].

In past papers [2, 7], studies about the structure of Gröbner bases and of standard pairs for the minimum cost flow problem have been done. The lower bound 1 of arithmetic degree and the upper bound $\frac{1}{d}\binom{2(d-1)}{(d-1)}$, d is the number of vertices, is shown in [7], using the following two results, one is about the characterization of Gröbner basis [6] and the other is about the special hypergeometric function [2].

Now we focus on dual problems of the minimum cost flow problems on acyclic tournament graph based on those studies. Duality of problem has some interesting properties:

- If feasible solutions exist both in primal and dual, the values of the objective functions correspond to each other.
- Circuits of graph of primal problem associate with cutset of graph of dual problem.
- Dual problems of "dual problems" return to primal problems.

In this paper, we investigate Gröbner bases and standard pairs of dual problems, using TiGERS and Macaulay 2. By the result, it is assumed that arithmetic degree has exponential order even in minimum case.

This paper is organized as follows. In Chapter 2 we introduce an adequate term order for cost vector which has negative elements and find Gröbner basis with respect to the ordering, and additionally find universal Gröbner basis which is based on all possible term order. In Chapter 3 we analyze standard pairs associated with Gr̈obner bases found in Chapter 2 and evaluate the arithmetic degree. In Chapter 4 we conclude this paper.

2 Toric Ideals of Minimum Cost Flow Problem

First we give a definition about integer programming problem. The form of primal problem is as follows:

$$\begin{array}{ll} \text{min.} & \boldsymbol{c}\cdot\boldsymbol{x} \\ \text{sub. to} & A\boldsymbol{x}=\boldsymbol{b}, \boldsymbol{x}\geq 0 \\ & A\in\mathbb{Z}^{d\times n}, \boldsymbol{x}\in\mathbb{N}^n, \boldsymbol{c}\in\mathbb{R}^n, \boldsymbol{b}\in\mathbb{Z}^d \end{array}$$

Now we consider minimum cost flow problem on acyclic tournament graph, so we let A be incidence matrix, b be the sum of incoming / outgoing flow, and c be a cost of each vertex. And variable x means actual quantity of flow. In this form, d representes the number of vertices and n the number of edges. Then $n=\binom{d}{2}$.

Next we introduce term orders and toric ideals. We define a monomial $\boldsymbol{x}^\alpha$ for $\boldsymbol{x}=(x_1,\cdots,x_n)$ and $\alpha=(a_1,\cdots,a_n)\in\mathbb{Z}^n$ as follows:

$$\boldsymbol{x}^\alpha = x_1^{\alpha_1},\cdots,x_n^{\alpha_n}$$

Let $\succ$ be a total order on monomial $K[x_1,\cdots,x_n]$, as K is a field. $\succ$ is called a term order when $\succ$ satisfies the following.

1. If $\boldsymbol{x}^\alpha\succ\boldsymbol{x}^\beta$, then $\boldsymbol{x}^\alpha\boldsymbol{x}^\gamma\succ\boldsymbol{x}^\beta\boldsymbol{x}^\gamma, for \forall\boldsymbol{x}^\gamma$

2. $\forall \boldsymbol{x}^\alpha \in K[x_1, \cdots, x_n] \backslash \{1\},\ \boldsymbol{x}^\alpha \succ 1$

In this study, we use term order generated by cost vector. Let cost vector $\boldsymbol{c}$ be non-negative. First, monomials are compared by inner products with vector $\boldsymbol{c}$. If the values are same, then compared by another term order.

Now we consider A as a set of column vectors $\{\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n\}$. Each $\boldsymbol{a}_i$ is identified with a monomial $\boldsymbol{t}^{a_i}$ in the Laurent polynomial ring $k[\boldsymbol{t}^{\pm 1}] := k[t_1, \cdots, t_d, t_1^{-1}, \cdots t_d^{-1}]$.

Definition 2.1 *Consider the homomorphism*
$$\pi : k[x_1, \cdots, x_n] \longrightarrow k[\boldsymbol{t}^{\pm 1}], \quad x_i \longmapsto \boldsymbol{t}^{\boldsymbol{a}_i}$$

The kernel of π is denoted I_A and called the toric ideal of A. In other words, I_A is like following[12]:

$$I_A = \langle \boldsymbol{x}^{\boldsymbol{u}^+} - \boldsymbol{x}^{\boldsymbol{u}^-} \mid \boldsymbol{u} \in \mathrm{Ker}(A) \cap \mathbb{Z}^n \rangle$$

Now we introduce Gröbner basis.

Definition 2.2 *Let I be an ideal on $K[x_1, \cdots, x_n]$ and $\succ$ be a term order. Now we define the initial ideal of I as*

$$in_\succ(I) := \langle in_\succ(f) :\ f \in I \rangle.$$

A finite set of polynomials $\mathcal{G} \subset I$ is called a Gröbner basis when for any $f \in I$ there exists $g_i \in \mathcal{G}$ such that $in(f)$ is divisible by $in(g_i)$. If a cost is positive, Gröbner basis always exists. And we introduce universal Gröbner basis.

Definition 2.3 *Universal Gröbner basis of an ideal I is the union of reduced Gröbner bases of I for all possible term orders.*

Consequently every ideal $I \subset K[x_1, \cdots, x_n]$ has a finite universal Gröbner basis.

$\boldsymbol{u}^+$ is the set of positive elements of $\boldsymbol{u}$ and $\boldsymbol{u}^-$ is that of negative elements of $\boldsymbol{u}$. We indicate the universal Gröbner basis of I_A as $\mathcal{U}_A$. For example, concerning $\boldsymbol{u} = (1, 0, 1, -1)$, toric ideal is $x_1x_3 - x_4$.

Lemma 2.4 *Toric ideals I_A are generated by finite binomials.*

2.1 Case of Primal Problem

First we consider on acyclic tournament graph with 4 vertices. Then an incidence matrix A is as follows.

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{pmatrix}$$

But one row is dependent on other rows, so we remove the lowest row which represents a sink. The kernel of A is a set of linear combination of $(1, -1, 0, 1, 0, 0)$, $(1, 0, -1, 0, 1, 0)$, and $(0, 1, -1, 0, 0, 1)$. So toric ideals of A is $\langle x_{12}x_{23} - x_{13}, x_{12}x_{24} - x_{14}, x_{13}x_{34} - x_{14}, x_{23}x_{34} - x_{24}, x_{13}x_{24} - x_{14}x_{23} \rangle$. The polynomials construct circuits of graph.

Second, we introduce initial terms. Let us assume the cost vector $\boldsymbol{c}$ is $(2, 1, 1, 2, 3, 1)$. Then the initial term of $x_{12}x_{23} - x_{13}$ is $x_{12}x_{23}$ because $2{+}2 > 1$. Similarly, $x_{12}x_{24}$ and $x_{13}x_{14}$ become initial terms.

2.2 Case of Dual Problem

2.2.1 Transformation of Problem

The form of dual problem is as follows:

$$\begin{array}{ll} \text{max.} & \boldsymbol{b} \cdot \boldsymbol{y} \\ \text{sub. to} & A^T \boldsymbol{y} \le \boldsymbol{c} \\ & A^T \in \mathbb{Z}^{n \times (d-1)}, \boldsymbol{y} \in \mathbb{Z}^{d-1}, \\ & \boldsymbol{c} \in \mathbb{R}^n, \boldsymbol{b} \in \mathbb{Z}^{d-1} \end{array}$$

We transform this as follows, as to deal it easily.

$$\begin{array}{ll} \text{min.} & (-\boldsymbol{b}\ \boldsymbol{0})\binom{\boldsymbol{y}}{\boldsymbol{z}} \\ \text{sub. to} & \begin{pmatrix} A^T & I_n \end{pmatrix} \begin{pmatrix} \boldsymbol{y} \\ \boldsymbol{z} \end{pmatrix} = \boldsymbol{c} \\ & A^T \in \mathbb{Z}^{n \times (d-1)}, \boldsymbol{y} \in \mathbb{Z}^{d-1}, \boldsymbol{c} \in \mathbb{R}^n, \\ & \boldsymbol{b} \in \mathbb{Z}^{d-1}, \boldsymbol{z} \in \mathbb{N}^n \end{array}$$

2.2.2 Universal Gröbner Basis and Circuits

Definition 2.5 *For a matrix A, a non-zero vector $u \in ker(A)$ is a circuit if* $\mathrm{supp}(u) := \{i : \boldsymbol{u}_i \ne 0\}$ *is minimal with respect to inclusion, and the elements of $\boldsymbol{u}$ are relatively prime.*

We define $\mathcal{C}_A$ as $\{x^{u+} - x^{u-} : \boldsymbol{u}$ is a circuit of $A\}$, and represents universal Gröbner basis as $\mathcal{U}_A$.

Definition 2.6 (unimodularity) *We define a non-singular matrix whose partial $d \times d$ determinant has absolute value 1 an unimodular matrix. Moreover, if determinant of any non-singular submatrix is 0 or ± 1, the matrix is called totally unimodular.*

Lemma 2.7 *If A is a totally unimodular matrix, $\mathcal{C}_A = \mathcal{U}_A$.*

In the case of minimum cost flow problems, the incidence matrices of primal problem A is totally unimodular, so that of dual problem $(A^T I_n)$ is unimodular. Hence $\mathcal{C}_A = \mathcal{U}_A$ stands, and for any cost vector, reduced Gröbner bases is contained in $\mathcal{C}_A$.

Then let $\boldsymbol{x}_C$ be the sum of i-th column vector of $\begin{pmatrix} I_{d-1} \\ -A^T \end{pmatrix}$ for all $i \in C$: C is a set of vertices contained in cutset.

Theorem 2.8 *A basis of* $\mathrm{Ker}((A^T\ I_n))$ *consists of* $\{x_{\{1\}}, \cdots, x_{\{d-1\}}\}$. *In other words,* $\mathrm{Ker}((A^T\ I_n))$ *is a set of linear combinations of each column vector of following matrix* $\begin{pmatrix} I_{d-1} \\ -A^T \end{pmatrix}$.

Theorem 2.9 *The set of circuit of a matrix $(A^T\ I_n)$ is $\{\boldsymbol{x}_C \colon C \subseteq \{1, \ldots, d-1\}\}$.*

Proof

Let $\boldsymbol{a} \in \mathrm{Ker}((A^T, I_n)) \cap \mathbb{Z}^{d-1+n}$(assume $\boldsymbol{a} \neq 0$). From Theorem 2.8, $\boldsymbol{a}$ is like as follows.

$$\begin{aligned}\boldsymbol{a} &= (a_1, \cdots, a_{d-1}, a_{12}, \cdots, a_{1d}, a_{23}, \cdots, a_{d-1,d}) \\ &= \sum_{i=1}^{d-1} k_i x_{\{i\}},\ k_i \in \mathbb{N}\end{aligned}$$

We can suppose $k_1 \neq 0$ without loss of generality.

First we assume that some $k_j \neq 0$ ($j = 2, \cdots, d-1$) is not equal to k_1 and show a contradiction.

We assume $C = \{i \in \{1, \ldots, d-1\} \colon k_i = k_1\}$. Then $a_p = k_p = k_1$ for any $p \in C$. And for any $p \in C$ and $q \notin C$, if $p < q$ then $a_{pq} = -k_p + k_q \neq 0$, otherwise $a_{qp} = k_p - k_q \neq 0$. Therefore $\mathrm{supp}(\boldsymbol{a}) \supsetneq \mathrm{supp}(\boldsymbol{x}_C)$ stands, thus $\boldsymbol{a}$ is not circuit.

So for any $j = 2, \cdots, d-1$, $k_j = k_1$ or $k_j = 0$. Then $\boldsymbol{a} = k_1 x_{C'}(C' = \{j\ ;\ k_j = k_i\})$ and $\mathrm{supp}(\boldsymbol{a}) = \mathrm{supp}(x_{C'})$.

Next, we consider for cutsets $C_1, C_2 \subseteq \{1, \ldots, d-1\}$ which is $C_1 \neq C_2$,

1. The case of $C_2 \subseteq C_1$

If we take $p \in C_1 - C_2, q \in C_2$, the edge (p, d) is contained in the cutset by C_1 but is not contained in that by C_2. So $(\boldsymbol{x}_{C_1})_{pd} \neq 0$, and $(\boldsymbol{x}_{C_2})_{pd} = 0$. Moreover, the edge (p, q) is contained in the cutset by C_2 but is not contained in that by C_1. So $(\boldsymbol{x}_{C_1})_{pq} = 0$, and $(\boldsymbol{x}_{C_2})_{pq} \neq 0$. Consequently, there is no relation of inclusion between the support of x_{C_1} and x_{C_2}.

2. The case that C_1 and C_2 have no relation of inclusion each other

By taking such $p \in C_1 - C_2, q \in C_2 - C_1$ and considering the edges (p, d) and (q, d), we can show the independency between the support of x_{C_1} and x_{C_2}, as a same way as the case $C_2 \subseteq C_1$. □

2.2.3 Gröbner Bases when Cost Vector is Negative

First we need to settle an adequate term order for negative cost vector. The elements of the cost vector $(-\boldsymbol{b}\ 0)$ can be positive and negative, so we cannot decide term orders simply according to the order of the elements of the cost. Then we introduce new non-negative vector, keeping generated initial terms unchanged.

Lemma 2.10 ([12]) *For a cost vector ω, when non-negative vector ω' exists and $in_\omega(I) = in_{\omega'}(I)$, ω compose a term order and the Gröbner bases for ω is equal to that for ω'.*

Gröbner basis generates ideal I. Thus if we can show a non-negative vector β exists when $Ax = b$ is feasible and $in_{(-\boldsymbol{b})}(g) = in_\beta(g)$ where g is any elements of Gröbner basis, we can compose a term order from cost vector $(-\boldsymbol{b}\ 0)$.

In primal problem $A\boldsymbol{x} = b$, one of feasible solution $\boldsymbol{f} = (f_{12}, \cdots f_{d-1,d})$ is non-negative. From it, we take β as

$$\beta = (0, \cdots, 0, \boldsymbol{f})$$

then for any vector $\boldsymbol{x}_C$,

$$\begin{aligned}\beta \cdot \boldsymbol{x}_C &= \sum_{i \notin C, j \in C, i<j} \beta_{ij} - \sum_{i \in C, j \notin C, i<j} \beta_{ij} \\ &= -\sum_{i \in C} b_i \\ &= (-\boldsymbol{b}) \cdot \boldsymbol{x}_C\end{aligned}$$

therefore $in_{(-\boldsymbol{b})}(g) = in_\beta(g)$, we can compose a term order from $(-\boldsymbol{b}, 0)$. Next we calculate a Gröbner basis.

Theorem 2.11 *When cost vector* $-\boldsymbol{b} = (b_1, \cdots, b_n)$ *satisfies* $-b_i < 0 (\forall i)$*, reduced Gröbner basis is* $\{-x_{\{1\}}, \cdots, -x_{\{d-1\}}\}$.

In other words, for each vertex, the set of the cutsets between one vertex and the others makes Gröbner basis.

Proof Universal Gröbner basis is a set of all cutsets. And the two sets of edges,

- Cutset edges from i when the largest index is i in cutset C, and
- Cutset edges from i when cutset $C = \{i\}$.

are equal. It means, for a certain cutset $\mathcal{K}$ (suppose the largest number is i), its initial term contains the monomial $\prod_{k=i+1}^{n} y_{ki} \cdots (1)$. On the other hand, when a cutset contains only i, an associated binomial is

$$\prod_{j=i+1}^{n} y_{ij} - x_i \prod_{j=1}^{i-1} y_{ji}$$

its initial term based on term order by cost vector $(-\boldsymbol{b}\ 0)$ is $\prod_{j=i+1}^{n} y_{ij} \cdots (2)$, so (2) can divide (1). $\square$

We calculated the reduced Gröbner bases using TiGERS. Then the minimum number of elements of reduced Gröbner basis is $d-1$.

3 Standard Pairs of Minimum Cost Flow Problem

First we give some definitions about standard pairs. For a fixed cost vector $\boldsymbol{c}$, let $O_{\boldsymbol{c}} \subset \mathbb{N}^n$ be the set of all the optimal solutions. Let N_c be the set of non-optimal solution on linear integer programming problems. Then N_c is $\mathbb{N}^n \backslash O_c$.For $\boldsymbol{a} \in \mathbb{N}^n$ and $\sigma \subseteq \{1, \cdots, n\}$, we define a set of points $(\boldsymbol{a}, \sigma)$ as $\{\boldsymbol{a} + \sum_{i \in \sigma} k_i e_i \mid k_i \in \mathbb{N}\}$. Then constraints are written as

$$\begin{aligned}A\boldsymbol{u} &= A(\boldsymbol{a} + \sum_{i \in \sigma} k_i e_i) \\ &= A\boldsymbol{a} + \sum_{i \in \sigma} k_i a_i \\ &= \boldsymbol{b}\end{aligned}$$

Consequently there are n equations, and the form is $\sum_{i \in \sigma} k_i a_i = \boldsymbol{b} - A\boldsymbol{a}$.

Lemma 3.1 ([10]) *The monomial which is not contained in* $in_C(I_A)$ *has one-to-one relation with* O_C.

Definition 3.2 $(\boldsymbol{a}, \sigma)$ *is a standard pair of* $O_{\boldsymbol{c}}$ *if*

1. $\mathrm{supp}(\boldsymbol{a}) \cap \sigma = \phi$.
2. $(\boldsymbol{a}, \sigma) \subseteq O_{\boldsymbol{c}}$
3. $(\boldsymbol{a}, \sigma) \not\subset (\boldsymbol{b}, \tau)$ *for any* $(\boldsymbol{b}, \tau)$ *which satisfies 1. and 2..*

We can obtain a set of standard pairs by a decomposition of O_c. The number of standard pairs is finite and unique. Now we call the number arithmetic degree. If the arithmetic degree is n, we can find an optimal solution by solving at most n equations [3].

For example, we consider the case of tournament graph of 3 vertices. Then A is

$$\begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix}.$$

Thus toric ideal I_A is kernel of A, in fact $\langle x_1 x_3 - x_2 \rangle$. There are two possible cases by value of $\boldsymbol{c}$.

1. The case when $c_1 + c_3 > c_2$

The initial term is x_1x_3. Thus $N_c = ((1,0,1) + \mathbb{N}^3)$, $O_c = \mathbb{N}^3 \backslash N_c$. Then standard pairs are $((0,0,0),(1,2))$ and $((0,0,0),(2,3))$ (See Figure 1). Thus the arithmetic degree is 2. By solving two cases $A \begin{pmatrix} x_{12} \\ x_{13} \\ 0 \end{pmatrix} = \boldsymbol{b}$ and $A \begin{pmatrix} 0 \\ x_{13} \\ x_{23} \end{pmatrix} = \boldsymbol{b}$, we can find an optimal solution. In the latter case, the flow is like as Figure 2.

2. The case when $c_1 + c_3 < c_2$

The initial term is x_2. Thus $N_c = ((0,1,0) + \mathbb{N}^3)$. Then standard pair is $((0,0,0),(1,3))$. The arithmetic degree is 1. By solving a case $A \begin{pmatrix} x_{12} \\ 0 \\ x_{23} \end{pmatrix} = \boldsymbol{b}$, we can find an optimal solution. Then flow is like as Figure 3. There is no flow at $(1,3)$.

Lemma 3.3 ([13]) *N_c can be written as the form of $\cup_{i=1}^{s}(p_i + \mathbb{N}^n)$. Such set p_i has a relation with a Gröbner basis.*

Theorem 3.4 ([4]) *If $N_c = \cup_{i=1}^{s}(p_i + \mathbb{N}^n)$ and $p_i \in \{0,1\}^n$ $(i = 1, \cdots, s)$, then any standard pair is the form of $((0,\cdots,0),\sigma)$.*

3.1 Standard Pairs in Primal Problem

In primal problem, minimum arithmetic degree is 1. Let us consider a cost vector such $c_{ij} > \sum_{i}^{j-1} c_{i,i+1}$ for any i, j s.t. $j > i+1$). Then the set of optimal solution O_c is $\sum_{i=1}^{d-1} k_i e_{i,i+1}$, which implies that $N_c = \cup_{j-i \geq 2}(e_{ij} + \mathbb{N}^n)$. Therefore arithmetic degree, which is the number of standard pair, is only one, and the form is as follows:

$$((0,\cdots,0),\{(1,2),(2,3),\cdots,(d-1,d)\})$$

On the other hand, maximum arithmetic degree is equal to $(d-1)$-th Catalan number $\frac{1}{d}\binom{2(d-1)}{d-1}$. The case is when cost vector $\boldsymbol{c}$ satisfies the following:

$$\begin{array}{lcll} c_{ij} + c_{jk} & > & c_{ik} & \text{(for any } i < j < k) \\ c_{ik} + c_{jl} & > & c_{il} + c_{jk} & \text{(for any } i < j < k < l) \end{array}$$

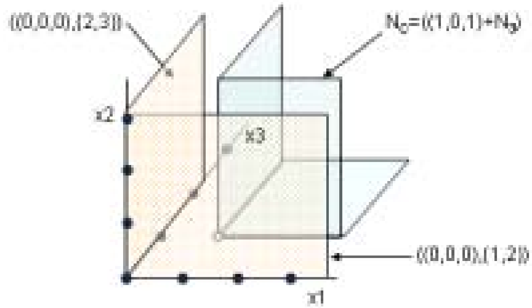

Figure 1: Space of N_c and O_c in the case $c_{12} + c_{23} < c_{13}$

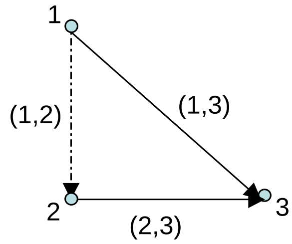

Figure 2: Flow in the case $c_{12} + c_{23} > c_{13}$

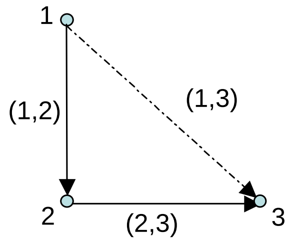

Figure 3: Flow in the case $c_{12} + c_{23} < c_{13}$

Then $N_c = \left(\cup_{i<j<k}((e_{ij} + e_{jk}) + \mathbb{N}^n)\right) \cup \left(\cup_{i<j<k<l}((e_{ik} + e_{jl}) + \mathbb{N}^n)\right)$.

Theorem 3.5 ([7]) *The number of spanning tree such*

- *does not contain both (i,k) or (j,k)*
- *does not contain both (i,k) or (j,l)*

is $\frac{1}{d}\binom{2(d-1)}{d-1}$ [11]. *Thus arithmetic degree increases exponential order for d in maximum case.*

3.2 Standard Pairs in Dual Problem

3.2.1 Standard Pairs by Universal Gröbner Basis

By Theorem 2.8, 2.9 and the definition of standard pair, each standard pair of toric ideal represents a choice of one edge from a set of flowing-out edges for cutset C (when the initial term is composed by only edges), otherwise a choice of either one point in C or a set of flowing-in edges for C (when the initial term includes vertices).

3.2.2 Case on Negative Cost Vector

We consider the case that all elements of $-\boldsymbol{b}$ is negative. From Theorem 2.11, the set of Gröber Bases is $\{-x_{\{1\}}, \cdots, -x_{\{d-1\}}\}$. So the set of initial terms is as follows:

$$\left\{ \prod_{j=i+1}^{d} y_{ij} \;\middle|\; 1 \leq i \leq d-1 \right\}.$$

Then

$$\begin{aligned} N_c =& ((\underbrace{1, \cdots, 1}_{y_{12},\cdots,y_{1d}}, 0 \cdots, 0) + \mathbb{N}^n) \cup \\ & ((0, \cdots, 0, \underbrace{1, \cdots, 1}_{y_{23},\cdots,y_{2d}}, 0 \cdots, 0) + \mathbb{N}^n) \cup \cdots \cup \\ & ((0, \cdots, 0, 1) + \mathbb{N}^n). \end{aligned}$$

So standard pairs are the set of the following form:

$$(\mathbf{0}, (\underbrace{0, \cdots, 1, 0,}_{\text{just one of } y_{12}\cdots y_{1d} \text{ is } 1} \underbrace{0, \cdots, 1, 0,}_{y_{23}\cdots y_{2d}} \cdots)).$$

In this case, the arithmetic degree is $(d-1)!$. It corresponds to the number of feasible spanning trees. By non-zero elements in standard pair, we can compose a spanning tree.

Definition 3.6 ([1]) *Co-tree is a set of edges which is a complement of a spanning tree.*

Spanning tree is maximal set of edges which does not include circuits, and co-tree is maximal set of edges which does not include cutsets.

Theorem 3.7 ([8]) *Optimal solution of dual problem does not include a cutset.*

3.2.3 Max / Min Number of Arithmetic Degree

On basis of Gröbner bases found in Section 3.2.5, we calculated standard pairs for each initial term, using Macaulay 2.[5] The result is in Table 1.

d	arithmetic degree min	arithmetic degree max	$(d-1)!$	d^{d-2}
3	1	3	2	3
4	2	12	6	16
5	4	68	24	125
6	12	Too Large	120	1296

Table 1: Arithmetic degree

The minimum number indicates the case that the size of Gröbner basis is minimum and the initial term is a term which the degree is less than that of the other. (e.g. for the ideal $y_{13}y_{14}y_{15} - x_3y_{12}$, the degree of x_3y_{12} is 2 and less than that of the other). So there are $1 \cdot 2 \cdots [\frac{n}{2}] \cdots 2 \cdot 1$ pairs.

On the other hand, it is obvious that the maximum number of arithmetic degree is less than d^{d-2}, which is the number of all spanning trees (i.e. the number of co-trees). But specific meaning of the number is not found yet.

4 Conclusion

We wrote integer programming problems as standard form, and assured that toric ideals by $(A^T I_n)$ is represented by an independent set of linear combination of each column vector of $\begin{pmatrix} I_n \\ -A^T \end{pmatrix}$, and they compose circuits.

And a certain cost vector can be replaced by non-negative vector, without changing generated $in(g)$. This made it possible that we composed a term order from any cost vector. So as an example, for a negative vector, the Gröbner basis was each column vector of $\begin{pmatrix} I_n \\ -A^T \end{pmatrix}$. Next, by calculating universal Gröbner basis by TiGERS, it was found that minimum size of Gröbner basis is $d-1$, and such Gröbner basis actually exists. Additionally, it was shown that all binomials in universal Gröbner basis compose circuit.

About standard pairs, focusing on the case that cost vector is negative, arithmetic degree is $(d-1)!$. It is equal to the number of feasible spanning trees, additionally there are one-to-one relations between each standard pair and feasible spanning tree.

And as a result of the experiment for universal Gröbner basis, it is conjectured that the arithmetic degree has exponential order for the number of vertices, even in the case which the size of Gröbner basis is minimum.

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