Geometric Shellings of 3-Polytopes*

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Abstract

A total order of the facets of a polytope is a geometric shelling if there exists a combinatorially equivalent polytope in which the corresponding order of facets becomes a line shelling. The subject of this paper is (geometric) shellings of 3-polytopes. Recently, a graph theoretical characterization of geometric shellings of 3-polytopes were given by Holt & Klee and Mihalisin & Klee.

- We first give a characterization of shellings of 3-polytopes.

Then we show sufficient conditions for a shellings to be geometric:

- the first and the last facet being adjacent,
- any facet (except the first two) being adjacent to no less than two previous facets or
- the induced orders being geometric shellings for two smaller polytopes made by dividing the polytope at a triple of facets adjacent to each other but not sharing a vertex.

Simple 3-polytopes allow perturbations of facets, thus may have more chance a shelling is geometric. As sufficient conditions for this case we show:

- the induced order being a geometric shelling for a smaller polytope made by removing a triangular or a quadrilateral facet or joining two consecutive facets in a shelling or
- the polytope only having triangular or quadrilateral facets.

A nongeometric shelling of a (simplicial) 3-polytope was first shown by Smilansky.

- We show such example for a simple 3-polytope, which is minimal with respect to the number of facets.

The discussions proceed in the polar setting: as total orders of vertices of the polar polytope. All of our main results can be stated in graph theoretical terms. The proof techniques used are elementary topology, graph theory, network flows and local changes of polytopes.

1 Introduction

A total order of the facets of a polytope is a *shelling* if it satisfies some topological condition (defined below at (∗)). Shellings have many applications both in combinatorial and computational geometry: for example, they are crucial for the upper bound theorem [1] [7], and are used in convex hull construction [10]. A total order of the facets of a polytope corresponds to a total order of the vertices of the polar polytope. Such order of the vertices is a *polar shelling*. A *line shelling* is some special shelling, and its polar becomes an ordering of the vertices by a sweep of hyperplanes,

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which we call a *polar line shelling* (see [4] [11] [12]). Polar line shellings are relevant to simplex methods in linear programming.

A total order of the facets of a polytope is a *geometric shelling* if in some combinatorially equivalent polytope the corresponding order becomes a line shelling. We call the total order of the vertices of the polar a *polar geometric shelling*. If a (polar) shelling is not a (polar) geometric shelling, it is called a (*polar*) *nongeometric shelling*.

In this paper, we discuss combinatorial properties of shellings of 3-polytopes. The face lattice of a 3-polytope is completely determined by its graph. Most of our discussions proceed in the polar setting: as total orders of the vertices of (a graph of) a 3-polytope. The main results of this paper can be described in purely graph theoretical terms. Holt & Klee [6] and Mihalisin & Klee [8] recently gave a characterization of polar geometric shellings in terms of directed graphs. Their results are used in some of our proofs.

Special cases of our interests are shellings of simple 3-polytopes (polar shellings of simplicial 3-polytopes). Such polytopes do not change their combinatorial properties under perturbation of facets (vertices). So, there may be more chance shellings (polar shellings) of such polytopes become geometric shellings (polar geometric shellings).

We give sufficient conditions or reductions for a total order of the facets (vertices) of a 3-polytope to be a geometric shelling (polar geometric shelling). Some of the results are for the special case of simple (simplicial) 3-polytopes.

All graphs considered are simple (i.e. no loops or multiple edges). Connectivity means vertex connectivity.

The following is basic for graphs of 3-polytopes.

Lemma 1.1
- (Steinitz' theorem) A graph is a graph of a 3-polytope if and only if it is planar and 3-connected.
- A graph is a graph of a simplicial 3-polytope if and only if it is maximal planar and has no less than four vertices.

Now to directed graphs. A total order of the vertices of a graph induces a directed graph by directing $\overrightarrow{st}$ for an edge $\{s,t\}$ with $s < t$. Directed graphs by such directing are acyclic. The symbol k will be used for the cardinality of the vertices, and the vertices will be labeled $1 < \cdots < k$ according to the total order.

We define a total order $F_1, \ldots, F_k$ of the facets of a 3-polytope to be a *shelling* if

$$\bigcup_{j=1}^{i} F_j \cong B^2 \quad (1 \le i < k), \tag{$*$}$$

where $\cong B^2$ means homeomorphic to a 2-ball. A total order of the vertices of a 3-polytope is a *polar shelling* if the corresponding order of the facets of the polar polytope is a shelling. The face lattice of a 3-polytope is determined by its graph. The graph of the polar of a 3-polytope is the dual of the graph of the original polytope. Since the definition of shellings and polar shellings are topological, (polar) shellings can be defined for graphs of 3-polytopes. We have a simple characterization for polar shellings:

Proposition 1.2 A total order of the vertices of a graph of a 3-polytope is a polar shelling if and only if the induced directed graph has a unique source and a unique sink. (The source vertex 1, the sink vertex k.)

Proof. Section 2. □

Since (polar) geometric shellings of 3-polytopes are also combinatorial properties depending only on the face lattice and the total order of the facets (or vertices), we can define them for graphs of 3-polytopes. A total order of the vertices of a graph of a 3-polytope is a polar geometric shelling if and only if there exists a 3-polytope with the face lattice as the graph and the z-coordinates of the vertices sorted according to the total order. This observation will be used later.

A necessary and sufficient condition for a total order of the vertices of a graph of a 3-polytope to become a polar geometric shelling was given recently by Holt &Klee [6] and Mihalisin & Klee [8]:

Theorem 1.3 ([6] [8]) A total order of the vertices of a graph of a 3-polytope is a polar geometric shelling if and only if the induced directed graph has a unique source, a unique sink (the source vertex 1 and the sink vertex k) and three independent paths from 1 to k.

The paths in a directed graph should be monotone.

We define a total order $F_1, \ldots, F_k$ of the facets of a 3-polytope to be a *strong shelling* if it is a shelling and

$$F_i \cap \left(\bigcup_{j=1}^{i-1} F_j \right) \text{ is the union of no less than two edges} \quad (3 \leq i \leq k).$$

The corresponding order of the vertices of the polar is a *polar strong shelling*.

Now, our main results:

Theorem 1.4 The following conditions are sufficient for a polar shelling total order of the vertices of a graph G of a 3-polytope to be a polar geometric shelling.

(i) Vertices 1 and k are adjacent.

(ii) The total order is a polar strong shelling.

(iii) There exists a nonface 3-cycle in G and the induced total order for both/either side(s) of the graph according to Theorem 5.1 is a polar geometric shelling.

Furthermore, when G is a graph of a simplicial 3-polytope, we have the following sufficient conditions. ($1 < s < k$, $k > 4$ for (iv), (v), (vi).)

(iv) Let G' be the graph obtained by deleting a degree 3 vertex s. The induced total order of G' is a polar geometric shelling.

(v) Suppose s is a degree 4 vertex with adjacent vertices t, u not adjacent to each other, $t < s < u$ and s, t, u not forming a nonface 3-cycle. Let G' be the graph obtained by deleting s and adding the edge $\{t, u\}$. The induced total order of G' is a polar geometric shelling.

(vi) Suppose the only smaller vertex adjacent to vertex s ($s > 2$) is $s - 1$ (or the only larger vertex adjacent to vertex $s - 1$ is s), and there are no nonface 3-cycles including these two vertices. Let v, w be the vertices adjacent to both $s-1$ and s, contract the edge $\{s-1, s\}$ and delete parallel edges (say, delete $\{s, v\}, \{s, w\}$, leaving $\{s-1, v\}, \{s-1, w\}$). The induced total order of the graph G' thus obtained is a polar geometric shelling. The vertex previously forming the endpoints of $\{s-1, s\}$ is labeled $s - 1$ (or s).

(vii) The degree of G is at most 4.

A nongeometric shelling of a 4-polytope was known and one of a 3-polytope was given by Smilansky [5] [11]. Both of the examples are for simplicial polytopes. We give a nongeometric shelling of a simple 3-polytope with 8 vertices and 6 facets, which is minimal with respect to the number of facets.

The results are discussed in detail in the following sections. Section 3 is used in section 4, but other sections are independent. The sections containing the results above are:

2. the proof of Proposition 1.2

3. (i) [Theorem 3.5]

4. (ii) [Theorem 4.2]

5. (iii) [Theorem 5.1], (iv), (v), (vi), (vii) [Theorem 5.3]

6. an example of a nongeometric shelling [Example 6.1]

2 Characterization of polar shellings

Proof. **(Proposition 1.2)**

only if: Suppose the total order was a polar shelling. By the definition $(*)$ of shelling, any vertex $i > 1$ is adjacent to a smaller vertex. Thus the only source is 1. The reverse order of a shelling is also a shelling. So, similarly, the only sink is k.

if: Suppose the induced directed graph had a unique source and a unique sink. Suppose the condition $(*)$ for shelling was satisfied for $i = 1, \ldots, r-1$, but violated for r (> 1). If $A = F_r \cap (\bigcup_{j=1}^{r-1} F_j)$ is empty or a vertex, r is also a source, contradicting the assumption. Thus, A should consist of no less than two connected components. Take a polytope with facets $F_1, \ldots, F_k$ realizing the situation. There exists a Jordan arc in $\bigcup_{j=1}^{r} F_j$ having in each side (interior points of) some facet F_j $(j > r)$. Hence, there should be at least one sink in each side, contradicting the assumption.

□

The "if" proof is not valid for dimension larger than 3. Indeed, we have a counterexample in dimension 4. For this case, replace $\cong B^2$ in the definition $(*)$ of shelling by $\cong B^3$, homeomorphic to a 3-ball.

Example 2.1 The 4-polytope with vertices

$$\begin{aligned} p_1 &= (0\,0\,0\,0), \quad p_2 = (2\,0\,0\,0), \quad p_3 = (0\,6\,0\,0), \\ p_4 &= (1\,1\,2\,0), \quad p_5 = (1\,2\,3\,0), \quad p_6 = (0\,0\,0\,1) \end{aligned}$$

is made by coning p_6 to the 3-polytope with vertices $p_1, \ldots, p_5$. The total order of the facets $\langle p_1p_2p_4p_6\rangle, \langle p_1p_4p_5p_6\rangle, \langle p_1p_3p_5p_6\rangle, \langle p_2p_3p_5p_6\rangle, \langle p_2p_4p_5p_6\rangle, \langle p_1p_2p_3p_4p_5\rangle, \langle p_1p_2p_3p_6\rangle$ forms a Hamiltonian path (indeed, a Hamiltonian cycle), thus the induced directed graph has a unique source and a unique sink. However, the union of the first four facets is not homeomorphic to B^3.

We give another characterization for polar shellings of simplicial 3-polytopes:

Proposition 2.2 Let G^* be a graph of a simplicial 3-polytope, take its embedding into the plane, and regard this as a 2-dimensional simplicial complex $\mathcal{C}^*$. A total order $F_1^*, \ldots, F_k^*$ of the vertices of G^* is a polar shelling if and only if

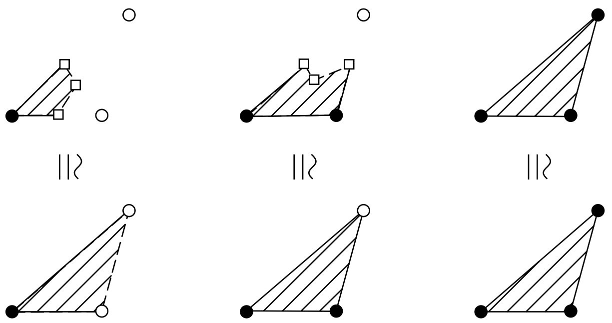
$$\bigcup_{F \in \bigcup_{j=1}^{i} \mathrm{star}(F_j^*)} \mathrm{int}(F) \;\cong\; \mathrm{int}(B^2) \quad (1 \leq i < k),$$

where $\mathrm{star}(F_j^*)$ is the star of F_j^*, $\mathrm{int}(F)$ and $\mathrm{int}(B^2)$ the interior of face F and 2-ball B^2.

Proof. Let $\mathcal{C}$ be the 2-dimensional simplicial complex defined by an embedding of G, the dual graph of G^*. The two simplicial complexes $\mathcal{C}$ and $\mathcal{C}^*$ are in dual (or polar) relation. Their stellar subdivisions $\mathrm{sd}(\mathcal{C})$ and $\mathrm{sd}(\mathcal{C}^*)$ are isomorphic. Denote the corresponding total order of the facets of $\mathcal{C}$ by $F_1, \ldots, F_k$. For $i = 1, \ldots, k$,

$$
\begin{aligned}
\mathrm{int}(\bigcup_{j=1}^{i} F_j) \quad &= \quad \bigcup \{\mathrm{int}(F) : F \in \mathcal{C},\, \mathrm{star}(F) \subset \bigcup_{j=1}^{i} \overline{F_j}\} \\
\underset{\text{stellar subdivision}}{=} \quad & \bigcup \{\mathrm{int}(\langle \sigma_1 \cdots \sigma_s \rangle) : \langle \sigma_1 \cdots \sigma_s \rangle \in \mathrm{sd}(\mathcal{C}),\, \sigma_1, \ldots, \sigma_s \in \mathcal{C},\, \sigma_1 \subsetneq \cdots \subsetneq \sigma_s, \\
& \qquad \mathrm{star}(\sigma_s) \subset \bigcup_{j=1}^{i} \overline{F_j} \text{ in } \mathcal{C}\} \\
\underset{\text{duality}}{\cong} \quad & \bigcup \{\mathrm{int}(\langle \sigma_s^* \cdots \sigma_1^* \rangle) : \langle \sigma_s^* \cdots \sigma_1^* \rangle \in \mathrm{sd}(\mathcal{C}^*),\, \sigma_s^*, \ldots, \sigma_1^* \in \mathcal{C}^*,\, \sigma_s^* \subsetneq \cdots \subsetneq \sigma_1^*, \\
& \qquad \overline{\sigma_s^*} \subset \bigcup_{j=1}^{i} \mathrm{star}(F_j^*) \text{ in } \mathcal{C}^*\} \\
\underset{(**)}{\cong} \quad & \bigcup \{\mathrm{int}(\langle \sigma_s^* \cdots \sigma_1^* \rangle) : \langle \sigma_s^* \cdots \sigma_1^* \rangle \in \mathrm{sd}(\mathcal{C}^*),\, \sigma_s^*, \ldots, \sigma_1^* \in \mathcal{C}^*,\, \sigma_s^* \subsetneq \cdots \subsetneq \sigma_1^*, \\
& \qquad \sigma_1^* \in \bigcup_{j=1}^{i} \mathrm{star}(F_j^*) \text{ in } \mathcal{C}^*\} \\
\underset{\text{stellar subdivision}}{=} \quad & \bigcup \{\mathrm{int}(F) : F \in \mathcal{C}^*,\, F \in \bigcup_{j=1}^{i} \mathrm{star}(F_j^*)\},
\end{aligned}
$$

where $\overline{F} = \{G : \text{ face of } F\}$. The homeomorphism $(**)$ can be made easily: the vertices in $\mathcal{C}^*$ are denoted by $\bullet$ for $F_1^*, \ldots, F_i^*$ and $\circ$ for the others.



Let i be smaller than k.

if: If $\bigcup_{j=1}^{i} F_j \cong B^2$ then $\mathrm{int}(\bigcup_{j=1}^{i} F_j) \cong \mathrm{int}(B^2)$.

only if: If $\bigcup_{F \in \bigcup_{j=1}^{i} \mathrm{star}(F_j^*)} \mathrm{int}(F) \cong \mathrm{int}(B^2)$ then we have $\mathrm{int}(\bigcup_{j=1}^{i} F_j) \cong \mathrm{int}(B^2)$. This implies that $F_1, \ldots, F_i$ are connected with respect to the relation being adjacent. If $\bigcup_{j=1}^{i} F_j \not\cong B^2$ then we can take a sequence of adjacent edges (i.e. a circuit) in $\bigcup_{j=1}^{i} \overline{F_j}$ where points not in $\bigcup_{j=1}^{i} F_j$ exist in both sides. Then there is a corresponding sequence of adjacent facets in $\bigcup_{j=1}^{i} \mathrm{star}(F_j^*)$ where points not in $\bigcup\{\mathrm{int}(F) : F \in \bigcup_{j=1}^{i} \mathrm{star}(F_j^*)\}$ exist in both sides. This contradicts $\bigcup\{\mathrm{int}(F) : F \in \bigcup_{j=1}^{i} \mathrm{star}(F_j^*)\} \cong \mathrm{int}(B^2)$.

$\square$

The proposition is not applicable for nonsimplicial cases or in dimension larger than three.

3 Network flows

The characterization by Holt & Klee and Mihalisin & Klee (Theorem 1.3) used "three independent paths from the smallest vertex 1 to the largest vertex k". The number of independent paths can be described in terms of network flows:

Lemma 3.1 For a directed graph and its vertices s and t,

(the maximum size of flows from s to t in the network of the directed graph with edge and vertex capacity 1)
= (the maximum cardinality of a set of independent paths from s to t in the directed graph)

Networks with both edge and vertex capacity 1 are not easy to handle directly. However, such cases can be reduced easily to the case only with edge capacity 1 condition. Let the original directed graph be G. Make another directed graph G' splitting any vertex v ($\neq s, t$) with indegree and outdegree both larger than 1 into v' and v'', and taking instead of the edges connected to v,

$$\{\overrightarrow{xv'} : \overrightarrow{xv} \in G\} \cup \{\overrightarrow{v''x} : \overrightarrow{vx} \in G\} \cup \{\overrightarrow{v'v''}\}.$$

Lemma 3.2 For a directed graph G, its vertices s and t, and the directed graph G' made as above,

(the maximum size of flows from s to t in the network of G with edge and vertex capacity 1)
= (the maximum size of flows from s to t in the network of G' with edge capacity 1)

The flows in networks with edge capacity 1 have good characterization (see, for example [9]):

Lemma 3.3 (Max flow-min cut theorem) For a network with a source s and a sink t,

(the maximum size of flows from s to t)
= (the minimum size of cuts separating s and t).

Furthermore, when the capacity of the edges are 0/1, a maximum flow with 0/1 entries exists.

The splitting above obviously causes splitting of the underlying undirected graph. For an undirected graph G, we split a vertex v by replacing it with v' and v'', connecting vertices adjacent to v either to v' or v'' and adding an edge $\{v', v''\}$. The following can be checked easily:

Lemma 3.4 Let v be a degree ≥ 4 vertex in a 3-connected graph. If we split v into v' and v'' so that both v' and v'' have degree 3 at least, the resulting graph is also 3-connected.

Combining these lemmas, we get a rather surprising result:

Theorem 3.5 If a total order of the vertices of a graph G of a 3-polytope is a polar shelling and the vertices 1 and k are adjacent, it is a polar geometric shelling.

Proof. Since the total order is a polar shelling, vertex 1 is the unique source and k is the unique sink in the induced directed graph, which we also denote by G.

As above Lemma 3.2, split each vertex s of G other than 1 or k, having indegree and outdegree both larger than 1. Label the new vertices $s' := s - 0.1$, $s'' := s + 0.1$. The directing of the splitted directed graph G' becomes equal to the one induced by the total order of this new labeling. It can be checked that G' is also planar and 3-connected (Lemma 3.4), 1 and k are the unique source and the unique sink and they are connected by an edge. The existence of independent paths from 1 to k is the same for G and G'.

Suppose three independent paths from 1 to k did not exist in G. By Lemmas 3.1, 3.2, 3.3, we can take a partition V, W of the vertices of G' with $1 \in V$, $k \in W$ and at most two edges directed from a vertex in V to a vertex in W.

The edge $\overrightarrow{1k}$ is directed from V to W. By 3-connectivity, there are at least 3 edges going out from vertex 1. Thus V contains at least two vertices. The same for W. Since G' is 3-connected, there are at least three edges between V and W, thus an edge $\overrightarrow{cd}$, $c \in W$, $d \in V$, $1 < c < d < k$ should exist. (The labels of the vertices can be fractionals.) Since c is not a source, there should exist an edge $\overrightarrow{ab}$, $a \in V$, $b \in W$, $1 \leq a < b \leq c$. Since d is not a sink, there should exist an edge $\overrightarrow{ef}$, $e \in V$, $f \in W$, $d \leq e < f \leq k$. We found three different edges $\overrightarrow{1k}$, $\overrightarrow{ab}$ and $\overrightarrow{ef}$ from V to W, a contradiction. □

Remark 3.6 The theorem is not true in dimension 4. Smilansky's treatment [11] of a polytope in Grünbaum & Sreedharan's list [5] shows a nongeometric shelling of a simplicial 4-polytope with the first and the last facet adjacent.

4 Polar strong shellings

Lemma 4.1 Let $1, \ldots, k$ be a total order of the vertices of a graph. Suppose vertices $1, 2$ are adjacent and any vertex s (> 2) is adjacent to at least two vertices among $1, \ldots, s-1$. Then for any vertex s (> 2), there exist two independent paths from 1 to s in the induced directed graph.

Proof. Induction on s. The claim is true for $s = 3$. Suppose it was true for $3, \ldots, s-1$. Case analysis by vertices adjacent to s:

- Vertices 1 and 2 are adjacent to s. Clear.
- Vertex 1 is adjacent to s, but 2 not. Suppose vertex $(2 <)$ p $(< s)$ was adjacent to s. There should exist two independent paths from 1 to p. Connecting one of them with edge $\overrightarrow{ps}$ makes a path from 1 to s. The edge $\overrightarrow{1s}$ makes another path.
- Vertex 2 is adjacent to s, but 1 not. Suppose vertex $(2 <)$ p $(< s)$ was adjacent to s. There should exist two independent paths from 1 to p. No matter whether 2 is on these paths or not, we can take a path from 1 to s with edges $\overrightarrow{12}$, $\overrightarrow{2s}$ and another one with edges from 1 to p and the edge $\overrightarrow{ps}$.
- Both of the vertices $1, 2$ are not adjacent to s. Suppose vertices $(2 <)$ $p < q$ $(< s)$ were adjacent to s. There should exist two independent paths from 1 to q. If p is on these paths, we can make two independent paths from 1 to s using edges $\overrightarrow{ps}$ and $\overrightarrow{qs}$. Suppose p was not on these paths. Consider the directed subgraph with these two independent paths, two independent paths from 1 to p and edges $\overrightarrow{ps}$, $\overrightarrow{qs}$.

  Now, split some vertices as before Lemma 3.2. (Vertices, $1, p, q, s$ are not splitted.) Consider the network with edge capacity 1. The followings are equivalent: (1) there were two independent paths from 1 to s in the graph before splitting, (2) there is a flow of size two from 1 to s and (3) any cut separating 1 and s has size at least two.

  Finally, we show (3). Let V, W, $1 \in V$, $k \in W$ be a partition of the vertices corresponding to a cut. If $p, q \notin W$, edges $\overrightarrow{ps}$, $\overrightarrow{qs}$ contributes two to the size of the cut. If $p \in W$, the cut separates 1 and p. Since there were two independent paths from 1 to p in the original graph, the cut has size at least two. The case $q \in W$ is similar.

□

Theorem 4.2 If a total order of the vertices of a graph G of a 3-polytope is a polar strong shelling, it is a polar geometric shelling.

Proof. Split some of the vertices of the induced directed graph as before Lemma 3.2, and relabel the vertices as in the proof of Theorem 3.5. (Vertices $1, 2, k$ are not splitted.) Consider the network with edge capacity 1. By Lemmas 3.1, 3.2, 3.3, the total order is a polar geometric shelling if and only if any cut separating 1 and k has size at least three. We want to show this indeed is the case.

Let V, W, $1 \in V$, $k \in W$ be a partition of the vertices corresponding to a cut. If $\#V \leq 2$, the cut has size at least three, because vertex 1 has degree ≥ 3 and there is no sink among the vertices in V. The case $\#W \leq 2$ can be shown similarly.

Now suppose $\#V, \#W \geq 3$. Analyze the cases by the types of vertices in W. Remind that a vertex $i \in \mathbb{Z}_{>0}$ became $i - 0.1$ and $i + 0.1$ when splitted.

- Suppose there was a vertex of type i or $i - 0.1$ $(\neq 2, k), (i \in \mathbb{Z}_{>0})$ in W. Let x be the smallest of such vertices. By Lemma 4.1, in the original directed graph there were two independent paths from 1 to the vertex which became x. Thus there still are two corresponding independent paths in the splitted graph. Hence there are at least two edges e, f from V to W. If there are more, we are done. Remark that all vertices in V are smaller than x. Otherwise, we can find a third edge from V to W, because there are no sinks in V.

  Since the graph is 3-connected, there should exist edges other than e, f between V and W. If some of them are directed from V to W, the cut has size at least three. Suppose not. For any edge from W to V, the terminating vertex should be smaller than x. Thus, the origin vertex is also smaller than x. Hence it should be of type $j + 0.1$ $(j \in \mathbb{Z}_{>0})$ or 2. In either case, there is a unique edge having this as a terminating vertex, and it should be an edge from V to W, thus e or f. Now we have shown that the vertices in W adjacent to vertices in V are the two terminating vertices of the edges e, f. This contradicts 3-connectivity.
- Suppose all vertices in W were of type $i + 0.1$ $(i \in \mathbb{Z}_{>0})$ or $2, k$. A vertex of type $i + 0.1$ or 2 is a terminating vertex of a unique edge, and the edge should be coming from V. Vertex k has degree 3. Thus, whether $\#W$ is 3 or more, the size of the cut is at least three.

□

We give another proof for the case of simplicial 3-polytopes. This is more geometric.

Proof. (for graphs of simplicial 3-polytopes)
First, a claim.
Claim: We can make a non-crossing straight line embedding of G into the plane satisfying the following conditions.

- By viewing the triangular regions as triangles, the graph can be regarded as a triangulation Δ of a triangle.
- For any $i \geq 3$, $\bigcup_{\sigma \in \Delta:\text{vertices of } \sigma \subset \{1,\ldots,i\}} |\sigma|$, the realization (union of faces as point sets) of the subcomplex of Δ induced by vertices $1, \ldots, i$, is a convex polygon $(***)$.

Proof of claim: Denote the embedding of the vertex i by $\boldsymbol{v}_i$. Choose three points not included in a line as $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$. We will place $\boldsymbol{v}_4, \ldots, \boldsymbol{v}_k$ in this order. Suppose $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{j-1}$ could be placed satisfying the conditions in the claim. The vertices adjacent to j are consecutive points $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_l$ of the boundary of the subcomplex of Δ induced by vertices $\{1, \ldots, j-1\}$. Otherwise, by considering the embedding of G into a 2-sphere (which is topologically unique by Whitney's theorem), we can see that there should exist more than one sinks, and the total order of the vertices would not be a polar shelling. The set of vertices adjacent to j becomes equal to the whole set of vertices in the boundary only when $j = k$.

The vertex $\boldsymbol{v}_j$ can be placed outside the realization of this subcomplex where the visible edges are $\{\boldsymbol{w}_1, \boldsymbol{w}_2\}, \ldots, \{\boldsymbol{w}_{l-1}, \boldsymbol{w}_l\}$. Such a point can always be chosen by performing a projective transformation on the subcomplex, if necessary. Since such projective transformations do not change the oriented matroid inside the subcomplex, $(***)$ is preserved for $i = 1, \ldots, j-1$. □

Now we make a simplicial 3-polytope showing the order is a polar geometric shelling. Lift $\boldsymbol{v}_i$ to $(\boldsymbol{v}_i, h_i)$, where $0 = h_1 \ll h_2 \ll \cdots \ll h_k$, and take their convex hull. This simplicial 3-polytope has Δ as the projection of its lower hull to $z = 0$ and G as its graph. The z-coordinates of the vertices are sorted according to the total order. □

Remark 4.3
- A strong shelling of a simplicial 3-polytope with the first and last vertex adjacent is equivalent to "a shelling order of vertices" in [2] [3].
- The only graph of a *simple* 3-polytope (i.e. planar, 3-connected and degree 3) which has a polar strong shelling is K_4.

Question 4.4 Can Theorem 4.2 be true for dimension ≥ 4 ?

5 Reductions

A nonface 3-cycle in a plane graph divides the graph into two sides, one inside and one outside. Both sides have the 3-cycle as their faces.

Theorem 5.1 Let $\{s, t, u\}$ be a nonface 3-cycle in a plane graph G. Fix a total order of the vertices of G for which the induced directed graph has a unique source and a unique sink (the source vertex 1 and the sink vertex k). There are three independent paths from 1 to k, if the same property holds for the graphs and their induced total orders of vertices as in the following.

- when $s, t, u \neq 1, k$ and $1, k$ are in different sides, both (1) the graph with the side including 1 with an extra new vertex "k" and edges $\overrightarrow{sk}$, $\overrightarrow{tk}$, $\overrightarrow{uk}$ and (2) the graph with the side including k with an extra new vertex "1" and edges $\overrightarrow{1s}$, $\overrightarrow{1t}$, $\overrightarrow{1u}$.
- when $s, t, u \neq 1, k$ and $1, k$ are in the same side, the graph with the side including $1, k$.
- when $s = 1$ and $t, u \neq k$ (this case is the same as $s = k$ and $t, u \neq 1$), the graph with the side including k.
- when $s = 1$, $t = k$, either of the two sides.

Proof. Easy. □

This theorem enables us to divide the graphs of 3-polytopes when showing polar geometric shelling.
For the rest, we restrict our attention to graphs of simplicial 3-polytopes.

Lemma 5.2 Let G be a graph of a simplicial 3-polytope with more than four vertices, $\{s, t\}$ its edge not included in a nonface 3-cycle, and v, w the vertices adjacent to both s and t. The graph G' obtained by contracting $\{s, t\}$ and deleting parallel edges (say, delete $\{t, v\}$, $\{t, w\}$, leaving $\{s, v\}$, $\{s, w\}$) is a graph of a simplicial 3-polytope with one less vertices.

Proof. Since edge $\{s, t\}$ is not included in a nonface 3-cycle, the double edges created when contracting $\{s, t\}$ are precisely the parallel edges deleted afterwards. Thus G' is simple, maximal planar and has no less than four vertices. □

The graph of a simplicial 3-polytope is unique for each case the number of vertices 4 or 5. It can be checked easily that every polar shelling of the vertices of such graphs is a polar geometric shelling (cf. Example 6.1).

Theorem 5.3 The following conditions are sufficient for a polar shelling total order of the vertices of a graph G of a simplicial 3-polytope to be a polar geometric shelling. ($1 < s < k$, $k > 4$ for (iv), (v), (vi).)

(iv) Let G' be the graph obtained by deleting a degree 3 vertex s. The induced total order of G' is a polar geometric shelling.

(v) Suppose s is a degree 4 vertex with adjacent vertices t, u not adjacent to each other, $t < s < u$ and s, t, u not forming a nonface 3-cycle. Let G' be the graph obtained by deleting s and adding edge $\{t, u\}$. The induced total order of G' is a polar geometric shelling.

(vi) Suppose the only smaller vertex adjacent to vertex s ($s > 2$) is $s-1$ (or the only larger vertex adjacent to vertex $s-1$ is s), and there are no nonface 3-cycles including these two vertices. Let v, w be the vertices adjacent to both $s-1$ and s, contract the edge $\{s-1, s\}$ and delete parallel edges (say, delete $\{s, v\}, \{s, w\}$, leaving $\{s-1, v\}, \{s-1, w\}$). The induced total order of the graph G' thus obtained is a polar geometric shelling. The vertex previously forming the endpoints of $\{s-1, s\}$ is labeled $s-1$ (or s).

(vii) The degree of G is at most 4.

Proof.

(iv), (v), (vi): By Lemma 5.2, the graph G' obtained after an operation among (iv), (v), (vi) is also a graph of a simplicial 3-polytope. Since we are assuming the induced total order of G' to be a polar geometric shelling, we can make a simplicial 3-polytope P' with G' its graph and the z-coordinates of the vertices sorted according to the total order. We want to show that a new vertex s (and $s-1$) can be added to P' making a polytope P whose graph is G with the z-coordinate of the new vertex in accordance with other vertices.

(iv) Place s just outside the facet of P' spanned by the three vertices originally adjacent to s in G. This will be the only facet visible form s. The polytope with this new vertex added to P' will be P. We can choose the z-coordinate of s between the smallest and the largest of the z-coordinates of the adjacent three points.

(v) Place s just outside the edge $\{t, u\}$ of P'. The two facets of P' adjacent at $\{t, u\}$ will be those visible from s. The polytope with this new vertex added to P' will be P. We can choose the z-coordinate of s between the z-coordinates of t and u.

(vi) Choose supporting hyperplanes for edges $\{s-1, v\}$ and $\{s-1, w\}$ in P'. Their intersection is a line touching P' at vertex $s-1$. We choose two new vertices s, $s-1$ on the line near the current $s-1$ in the opposite sides. The polytope with new vertices added to P' will be P with G as its graph.

Since we have 1-dimensional freedom in choosing each of the hyperplanes, we have 2-dimensional freedom for the line, and can assume its z-coordinate not to be constant. Among the two new vertices, choose the one with larger z-coordinate to be s and the other $s-1$. The one adjacent to vertices larger than s in the graph G (thus corresponding to the original s in G) is the larger one.

The rest of the proof depends on which of the conditions in the first statement was satisfied. Suppose the only smaller vertex adjacent to vertex s was $s-1$. If the vertex with smaller z-coordinate was adjacent to vertices larger than s in G, it becomes a local minimum in P when minimizing z, because all of its neighbouring vertices have larger z-coordinates. This contradicts the existence of the vertex 1 ($< s-1$) in P. The case the only larger vertex adjacent to vertex $s-1$ was s can be shown similarly.

(vii): Induction on the number of vertices. Cases with 4 or 5 vertices are true. We prove by case analysis.

- When there are nonface 3-cycles, all of them not including 1 or k, having a single vertex either 1 or k in one side of the 3-cycle. Dividing by such a 3-cycle does not decrease the number of vertices included, and we cannot apply induction. Since the degree is at most 4, exactly one such 3-cycle exists. Hence, there exists a vertex other than 1 or k not in the 3-cycle, and we can apply the reduction (iv) or (v) for this vertex.
- When there are nonface 3-cycles of other kind. We can divide the graph to the inner and outer side of the 3-cycle. In these subgraphs, the induced total order is a polar shelling, the degree of the vertices do not increase, and the vertices are less. By induction, the total orders of these graphs are polar geometric shellings. By Theorem 5.1, the total order of the original is also a polar geometric shelling.
- When there are no nonface 3-cycles. Apply reduction (iv) or (v) to a vertex other than 1 or k.

□

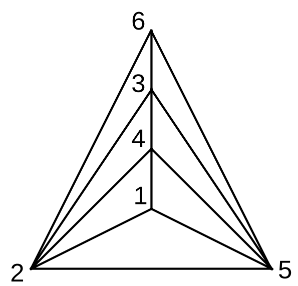
Remark 5.4 • By Euler's formula, graphs of simplicial 3-polytopes satisfying condition (vii) have six vertices at most. The unique combinatorial types of graphs of simplicial 3-polytopes with 4 or 5 vertices have degree at most 4. One type of the graphs of simplicial 3-polytopes with 6 vertices has degree at most 4. It can be checked directly that every polar shelling of the vertices of these three types of graphs is a polar geometric shelling.

- The condition being a graph of a *simplicial* 3-polytope is necessary for (iv), (v), (vi), (vii). Example 6.2 is a counterexample for (vii) without this condition of simpliciality.
- If we remove vertex $s = 1$ or k in (iv), (v), (vi), there is no guarantee we can add the vertex back to P' in accordance with the z-coordinate of the other vertices. In fact, examples in which local changes in P' as in the argument above does not work exist.

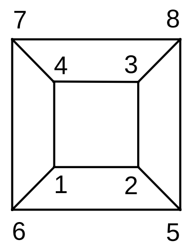
6 Examples

Example 6.1 The figure shows an example of a polar nongeometric shelling of a simplicial 3-polytope. Since the total order is defining a Hamiltonian path, it is a shelling. There are 6 vertices and 8 facets.

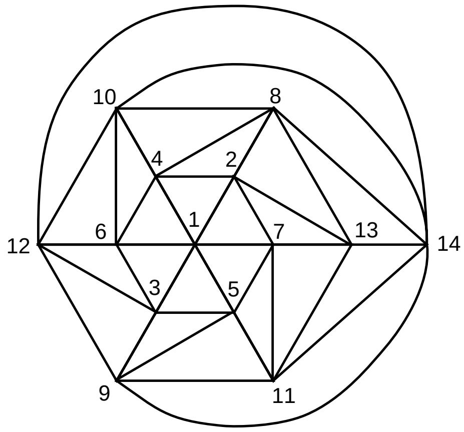
The graph of a 3-polytope with 4 vertices is K_4, thus of a simplicial 3-polytope. Any total order of its vertices is a polar geometric shelling. There are two combinatorial types of graphs of 3-polytopes with 5 vertices, one of a simplicial 3-polytope and the other not. Any polar shelling of the vertices of these graphs is a polar geometric shelling. Thus, this example is minimal with respect to the number of vertices. (There also exist other 3-polytopes with 6 vertices having polar nongeometric shellings. Though not simplicial, a prism has 5 facets which is the smallest among them.)



Example 6.2 In the polar setting, Smilansky's example [11] is a polar nongeometric shelling of a simple 3-polytope. It has 8 vertices and 6 facets.



Example 6.3 There is a polar geometric shelling of a graph of a simplicial 3-polytope which we cannot show by our sufficient conditions. (All vertices have degree ≥ 5, no edges $\{s-1, s\}$ $(2 < s < k)$ between consecutive vertices, no nonface 3-cycle.)



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