ANALYSIS OF GRÖBNER BASES FOR TORIC IDEALS OF ACYCLIC TOURNAMENT GRAPHS

by

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A Master's Thesis

修士論文

Submitted to

The Graduate School of Information Science

Faculty of Science

The University of Tokyo

on February 7, 2000

in Partial Fulfillment of the Requirements

for the Degree of Master of Science

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Title: Associate Professor of Information Science

ABSTRACT

Applications of Gröbner bases to some computationally hard problems in combinatorics using the discreteness of toric ideals have been studied in recent years. On the other hand, the properties of graphs may give insight into Gröbner bases. Although toric ideals of undirected complete graphs and bipartite graphs, which are homogeneous ideals, have been studied well, those of other graphs are not well understood. For the case of directed graphs, their universal Gröbner basis corresponds to the set of all the circuits of the graphs, but their toric ideals are not homogeneous with respect to ordinary grading. Thus toric ideals of directed graphs are interesting to study in graph theory. In this thesis, we analyze toric ideals of acyclic tournament graphs, which are the most fundamental directed graphs. We focus especially on the degree and the number of elements of its reduced Gröbner bases.

We first give the positive grading which makes the toric ideals homogeneous. We next give reduced Gröbner bases for toric ideals with respect to some term orders for both this grading and ordinary grading. We show that there exist term orders for which reduced Gröbner bases remain in polynomial order by characterizing reduced Gröbner bases in terms of circuits. Note that the universal Gröbner basis for these graphs is of exponential size.

We next analyze the number of elements and degree of reduced Gröbner bases with respect to various term orders. Generally the degree of reduced Gröbner bases for toric ideals is at most of exponential order, but in both cases of these two gradings, the degree remains in polynomial order since the matrix is unimodular. We are interested in how the number of elements can be bounded for the toric ideals of acyclic tournament graphs. Using properties of the cycle space of graphs, we show that the Gröbner bases we have given above are the examples achieving minimum number of elements or maximum degree in the case of ordinary grading. We next calculate for graphs with small number of vertices, and give upper bounds for the number of elements in the case of the grading whose toric ideal become homogeneous. We also analyze the number of elements for the purely lexicographic order.

We finally discuss applications to the minimum cost flow problem. Algorithms for integer programming using Gröbner bases have been studied recently, and those complexity depends on the size of the corresponding Gröbner basis. We apply our results using these algorithm to the minimum cost flow problem, analyze the complexity of algorithms and relate to the complexity of minimum mean cycle-canceling algorithm in minimum cost flow problem.

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Acknowledgements

I would like to thank Prof. Hiroshi Imai, who is my supervisor, and Dr. Kenichi Asai for their helpful advice. I would also like to thank members of the labratory, especially Mr. Fumihiko Takeuchi for his useful suggestion and discussion.

I would also like to thank Prof. Ryuuichi Hirabayashi and Prof. Yoshiko Ikebe who hold a private seminar. It is a great experience for me to discuss in this seminar.

I thank Mr. Hidefumi Ohsugi for his useful suggestion about Gröbner bases and toric ideals. I thank Mr. Masahiro Hachimori for his suggestion in some private seminars.

Finally, I thank my friends and my family for their encouragement.

Chapter 1

Introduction

1.1 Background

Recently, some algebraic approaches to many computationally hard problems in combinatorics have been studied. The main tool is the *Gröbner basis* for polynomial ideals, which is an important tool in computational algebra and algebraic geometry. Gröbner bases have provided new insight into some combinatorial problems such as integer programming [5, 8, 9, 21, 24], computational statistics [9], coding theory [13], and so on. The case of integer programming and computational statistics, *toric ideals* in a polynomial ring are also important tools. For example, Conti and Traverso [5] constructed an algorithm to solve integer programs using Gröbner bases via the discreteness of toric ideals. This algorithm have given insight into the structure of integer programming by associating reduced Gröbner bases with test sets in integer programming [21].

Related to some combinatorial problems in graph theory, toric ideals of graphs have been studied. De Loera, Sturmfels and Thomas [8] studied the toric ideals of undirected complete graphs and those Gröbner bases, and applied them to the triangulation of second hypersimplex and minimum weight perfect f-matching problem. Diaconis and Sturmfels [9] studied the toric ideals of bipartite graphs and those Gröbner bases, and applied them for sampling from conditional distributions and transportation problem. From the viewpoint of commutative algebra, Ohsugi and Hibi [16, 17] studied the toric ideals of general undirected graphs, and showed the conditions when the toric ideals are generated by quadratic binomials. Conversely, the properties of graphs may give insight into Gröbner bases.

The toric ideals of these two graphs are homogeneous with respect to the standard positive grading (i.e. the degree of all variables being 1). Homogeneous ideals have many good properties in commutative algebra theory and combinatorics. In particular, the *state polytope* [3], each of whose vertex corresponds to one reduced Gröbner basis of the ideal, can be defined, so all Gröbner bases of the ideal can be calculated by searching the edge graph of state polytope [12]. But toric ideals of other graphs are not well understood. It is because the toric ideals may not be homogeneous with respect to the standard positive grading.

1.2 Our objective

In this thesis, we study the toric ideals of acyclic tournament graphs, which are the most fundamental directed graphs. The toric ideals of acyclic tournament graphs are not homogeneous with respect to the standard positive grading. But they are homogeneous with respect to the specific positive grading which we call *graphic grading* in this thesis. In addition, since the vertex-edge incidence matrices of acyclic tournament graphs are unimodular, any elements in the toric ideals are square-free (i.e. each elements of exponent vector of each term is 0 or 1), and correspond to the circuits in the graphs. So we can characterize the reduced Gröbner bases of toric ideals with respect to some specific term orders in terms of circuits. We give the reduced Gröbner bases with respect to some purely lexicographic orders and degree lexicographic orders in both of the cases graphic grading and standard grading.

We focus especially on the degree and the number of elements in reduced Gröbner bases. Analysis of the Gröbner bases of acyclic tournament graphs are very important. Acyclic tournament graphs contains any acyclic tournament graphs as subgraphs, and undirected bipartite graphs can be regarded as the subgraphs of acyclic tournament graphs by directing each edge from one set of vertices in bipartite graphs to the other. By the elimination theorem(see [6]), reduced Gröbner bases of any subgraphs of acyclic tournament graphs can be obtained automatically if those of acyclic tournament graphs can be calculated. Thus the degree and the number of elements in reduced Gröbner bases of any subgraphs are less than those of acyclic tournament graphs. Thus the number of elements in reduced Gröbner bases of any subgraphs are less than those of acyclic tournament graphs. On the other hand, the number of elements in reduced Gröbner bases of graphs are related to the complexity of integer programming problems arising from the graphs.

The degree of general toric ideals are shown to be of exponential size by Sturmfels [19], but in the toric ideals of acyclic tournament graphs, since the vertex-edge incidence matrices are unimodular, the degree bound may be able to reduce. We show, in the case of standard grading, the degree becomes linear order.

The number of elements in reduced Gröbner bases for general homogeneous ideals are studied by Robbiano [18] using commutative ring theory. We show that the minimum number of elements in reduced Gröbner bases is $\binom{n}{2}-(n-1)=O(n^2)$. But the upper bound is not understood. To analyze the upper bound we calculate all reduced Gröbner bases for small n using TiGERS [11]. TiGERS is a software system implemented in C which searches the state polytope of a homogeneous toric ideal. We also consider the bound for the number of elements in reduced Gröbner bases with respect to the purely lexicographic orders. The reduced Gröbner bases with respect to the purely lexicographic orders are generally hard to compute, but they are independent of the positive grading. We implement the algorithm to check whether the Gröbner basis is the basis with respect to the purely lexicographic order [20], and analyze the upper bound.

We also study an application to the minimum cost flow problem. The minimum mean cycle-canceling algorithm [10] is known as a strongly polynomial time algorithm for minimum cost flow problems. In the main step of minimum mean cycle-canceling algorithm, the number of cycles which the algorithm may choose are all of the circuits and the number is of exponential. Using Conti-Traverso algorithm [5], the number of cycles which the algorithm may choose are at most the number of reduced Gröbner bases. Thus if we can bound the elements of reduced Gröbner bases, the number of cycles which may be chosen in minimum mean cycle-canceling algorithm may be reduced. But the complexity of Conti-Traverso algorithm is not known.

1.3 Organization of This Thesis

This thesis is organized as follows. In Chapter 2 we give some basic definitions of Gröbner bases and toric ideals. In Chapter 3 we deal with the reduced Gröbner bases of several term orders. We introduce standard grading and graphical grading, and give the reduced Gröbner bases of some term orders in terms of circuits of graphs. In Chapter 4 we analyze the degree and the number of elements in reduced Gröbner bases. For the graphical grading, we experiments on small graphs using TiGERS. In Chapter 5 we deal with the application to minimum cost flow problem. Finally in Chapter 6 we conclude this thesis.

Chapter 2

Preliminaries

In this chapter we give basic definitions of Gröbner bases and toric ideals. We refer to [6, 7] for the introduction of Gröbner bases, and [19, 20] for the introduction of toric ideals and their applications.

2.1 Gröbner Bases

Let k be a field and let $k[x_1, \dots, x_n]$ be the ring of polynomials in n variables with coefficients in k. A *monomial* in $k[x_1, \dots, x_n]$ is a product of powers of variables, i.e. $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$. We will write this monomial as $\mathbf{x}^{\mathbf{a}}$ where $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{N}^n$ is the vector of exponents. ($\mathbb{N}$ is the set of non-negative integer.) Hence the monomials in $k[x_1, \dots, x_n]$ are in bijection with the vectors in $\mathbb{N}^n$.

Definition 2.1 *$I \subseteq k[x_1, \dots, x_n]$, $I \neq \emptyset$ is an* ideal *if I satisfies the following:*

1. $f, g \in I \Longrightarrow f + g \in I$
2. $f \in I,\ h \in k[x_1, \dots, x_n] \Longrightarrow fh \in I$

We say that I is *generated by polynomials* $f_1, \dots, f_s$ and write $I = \langle f_1, \dots, f_s \rangle$ when for any $g \in I$, $g = \sum_{i=1}^{s} h_i f_i$ for some polynomials $h_1, \dots, h_s$. We say $f_1, \dots, f_s$ a *basis* for I.

Definition 2.2 *Let $m = \alpha \mathbf{x}^{\mathbf{a}} \in k[x_1, \dots, x_n]$ be a monomial. We define the* degree *of m with respect to a positive grading* $\deg(x_i) = d_i > 0$ *by*

$$\deg(m) = a_1 d_1 + a_2 d_2 + \cdots + a_n d_n.$$

Let $f = \sum_{i=1}^{s} \alpha_i \mathbf{x}^{\mathbf{a}_i} \in k[x_1, \dots, x_n]$*. We call* f *is* homogeneous *of degree* k *with respect to a positive grading* $\deg(x_i) = d_i > 0$ *if*

$$\deg(\alpha_1 \mathbf{x}^{\mathbf{a}_1}) = \deg(\alpha_2 \mathbf{x}^{\mathbf{a}_2}) = \cdots = \deg(\alpha_s \mathbf{x}^{\mathbf{a}_s}) = k.$$

Fix a positive grading $\deg(x_i) = d_i > 0$. For any polynomial $f \in k[x_1, \dots, x_n]$, we can write $f = f_0 + f_1 + \cdots + f_r$ such that each f_i is homogeneous of degree i. We call this r the degree of f and write $\deg(f)$.

Definition 2.3 *Let* I *be an ideal in* $k[x_1, \dots, x_n]$*. We call* I *a* homogeneous ideal *if, for any* $f = f_0 + f_1 + \cdots + f_r \in I$*, the homogeneous components* $f_0, f_1, \dots, f_r$ *are in* I*. Equivalently,* I *is a homogeneous ideal if* I *is generated by finite homogeneous polynomials* $g_1, \dots, g_s$*.*

Let M be the set of monomials in $k[x_1, \dots, x_n]$.

Definition 2.4 *Let* $\succ$ *be a total order on* M*. We call* $\succ$ *a* term order *on* M *if* $\succ$ *satisfies the following:*

1. $^{\forall}\mathbf{x}^{\alpha}, \mathbf{x}^{\beta}, \mathbf{x}^{\gamma} \in M$, $\mathbf{x}^{\alpha} \succ \mathbf{x}^{\beta} \Longrightarrow \mathbf{x}^{\alpha}\mathbf{x}^{\gamma} \succ \mathbf{x}^{\beta}\mathbf{x}^{\gamma}$.

2. $^{\forall}\mathbf{x}^{\alpha} \in M \setminus \{1\}$, $\mathbf{x}^{\alpha} \succ 1$.

For any term order $\succ$ and polynomial f, there exists the largest term with respect to the order in f. We say this term *initial term of* f and write $in_{\succ}(f)$, or short, $in(f)$.

Remark 2.5 *In this thesis, we line under the initial term of polynomial.*

We give some examples of term orders.

Definition 2.6 *Fix a variable ordering* $x_{i_1} \succ x_{i_2} \succ \cdots \succ x_{i_n}$*. We say* $\succ$ *is a* purely lexicographic order *induced by this variable ordering if, for any* $\mathbf{x}^{\alpha}$ *and* $\mathbf{x}^{\beta}$*,* $\mathbf{x}^{\alpha} \succ \mathbf{x}^{\beta}$ *if and only if there exists* $1 \le m \le n$ *such that* $\alpha_{i_k} = \beta_{i_k}$ *for* $k < m$ *and* $\alpha_{i_m} > \beta_{i_m}$*.*

Definition 2.7 *Fix a variable ordering* $x_{i_1} \succ x_{i_2} \succ \cdots \succ x_{i_n}$*. We say* $\succ$ *is a* degree lexicographic order *induced by this variable ordering if, for any* $\mathbf{x}^{\alpha}$ *and* $\mathbf{x}^{\beta}$*,*

$$\mathbf{x}^{\alpha} \succ \mathbf{x}^{\beta} \Longleftrightarrow \deg(\mathbf{x}^{\alpha}) > \deg(\mathbf{x}^{\beta}) \text{ or } \big(\deg(\mathbf{x}^{\alpha}) = \deg(\mathbf{x}^{\beta}) \text{ and } \mathbf{x}^{\alpha} \succ_{plex} \mathbf{x}^{\beta}\big)$$

($\succ_{plex}$ *is purely lexicographic order induced by* $x_{i_1} > x_{i_2} > \cdots > x_{i_n}$*.*)

Definition 2.8 *Fix a variable ordering* $x_{i_1} \succ x_{i_2} \succ \cdots \succ x_{i_n}$. *We say* $\succ$ *is a* degree reverse lexicographic order *induced by this variable order if, for any* $\mathbf{x}^\alpha$ *and* $\mathbf{x}^\beta$, $\mathbf{x}^\alpha \succ \mathbf{x}^\beta$ *if and only if*

- $\deg(\mathbf{x}^\alpha) > \deg(\mathbf{x}^\beta)$, *or*
- $\deg(\mathbf{x}^\alpha) = \deg(\mathbf{x}^\beta)$ *and there exists* m *such that* $\alpha_{i_j} = \beta_{i_j}$ *for* $j > m$ *and* $\alpha_{i_m} < \beta_{i_m}$.

Definition 2.9 *Let* $\omega \in \mathbb{R}^n_{\geq 0}$ *be a non-negative vector and* $\succ$ *be an arbitrary term order. We define a new term order* $\succ_\omega$ *as follows: for any* $\mathbf{x}^\alpha$ *and* $\mathbf{x}^\beta$,

$$\mathbf{x}^\alpha \succ_\omega \mathbf{x}^\beta \Longleftrightarrow \omega \cdot \alpha > \omega \cdot \beta \text{ or } (\omega \cdot \alpha = \omega \cdot \beta \text{ and } \alpha \succ \beta).$$

We say $\succ_\omega$ *a* refinement *of* ω *with respect to* $\succ$.

Definition 2.10 *A term order* $\succ$ *on* $k[x_1, \ldots, x_n, y_1, \ldots, y_m]$ *is an* elimination order with $\{x_1, \ldots, x_n\} \succ \{y_1, \ldots, y_m\}$ *if any monomial involving at least one of* $x_1, \ldots, x_n$ *is greater than all monomials in* $k[y_1, \ldots, y_m]$.

Example 2.11 *We consider a term order on the set of monomials in* $k[x, y]$ *with* $\deg(x) = \deg(y) = 1$. *If* $\succ$ *is the purely lexicographic order induced by the variable ordering* $x \succ y$, *then* $x^2 \succ xy^2$.

If $\succ$ *is a degree lexicographic order induced by the variable ordering* $x \succ y$, *then* $x^2 \prec xy^2$.

If $\succ$ *is a degree reverse lexicographic order induced by the variable ordering* $x \succ y$, *then* $x^2 \succ xy$.

If $\succ_{(3,1)}$ *is a refinement of* $(3, 1)$ *with respect to the purely lexicographic order induced by* $x \succ y$, *then* $x^2 \succ xy^2$. ■

Given a term order, we can define a *Gröbner basis* for an ideal with respect to the order.

Definition 2.12 *Let* I *be an ideal in* $k[x_1, \ldots, x_n]$ *and* $\succ$ *be a term order. We define the* initial ideal of I *as*

$$in_\succ(I) := \langle in_\succ(f) \colon f \in I \rangle.$$

Initial ideal of $I \subset k[x_1, \dots, x_n]$ can be drawn in the first quadrant of $\mathbb{N}^n$. For example, if $in_{\succ}(I) = \langle x^4y^2, x^3y^4, x^2y^5 \rangle$, then the exponent vectors of the monomials in $in_{\succ}(I)$ form the set

$$\left((4,2) + \mathbb{N}^2\right) \cup \left((3,4) + \mathbb{N}^2\right) \cup \left((2,5) + \mathbb{N}^2\right).$$

Thus we can draw this set as the set of integer points in the shaded area in Figure 2.1.

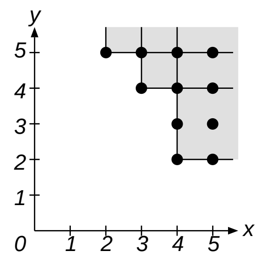

Figure 2.1: $in_{\succ}(I)$ can be drawn as a set of integer points in the first quadrant.

Definition 2.13 *Let I be an ideal in $k[x_1, \dots, x_n]$ and $\succ$ be a term order. A finite subset $\mathcal{G} = \{g_1, \dots, g_s\} \subset I$ is a* Gröbner basis for I with respect to $\succ$ *if, for any $f \in I$, there exists some $g_i \in \mathcal{G}$ such that $in_{\succ}(f)$ is divisible by $in_{\succ}(g_i)$.*

In other words, $\mathcal{G}$ is a Gröbner basis for I with respect to $\succ$ if its initial ideal $in_{\succ}(I)$ is generated by $\{in_{\succ}(g_i) \colon g_i \in \mathcal{G}\}$.

Definition 2.14 *Let I be an ideal in $k[x_1, \dots, x_n]$ and $\succ$ be a term order. A Gröbner basis $\mathcal{G} = \{g_1, \dots, g_s\}$ for I with respect to $\succ$ is* reduced *if $\mathcal{G}$ satisfies the following:*

1. *For any i, the coefficient of g_i is* 1.

2. *For any i, any term of g_i is not divisible by $in_{\succ}(g_j)$ $(i \neq j)$.*

Example 2.15 *Let $I = \langle x^2 + y^2 - 4, xy - 1 \rangle$ and $\succ$ be a purely lexicographic order induced by the variable ordering $x \succ y$. Then*

$$\mathcal{G} = \{x^2 + y^2 - 4, xy - 1, y^4 - 4y^2 + 1, x + y^3 - 4y\}$$

is a Gröbner basis for I but is not reduced.

$$\mathcal{G}' = \{y^4 - 4y^2 + 1, x + y^3 - 4y\}$$

is a reduced Gröbner basis for I. ■

Definition 2.16 *We define the degree of reduced Gröbner basis $\mathcal{G} = \{g_1, \ldots, g_s\}$ with respect to a positive grading $\deg(x_i) = d_i > 0$ as $\max_i \deg(g_i)$.*

We give some properties of Gröbner basis.

Proposition 2.17 *The reduced Gröbner basis is unique for an ideal and a term order.*

Proposition 2.18 *For any term order $\succ$, a Gröbner basis for I with respect to $\succ$ is a basis for I.*

Definition 2.19 *We define the* universal Gröbner basis *of I to be the union of reduced Gröbner bases of I with respect to all term orders.*

Universal Gröbner bases were introduced in [25].

Although there are infinite term orders, a universal Gröbner basis is finite.

Proposition 2.20 ([25]) *Every ideal $I \subset k[x_1, \ldots, x_n]$ has a finite universal Gröbner basis.*

We can define "division" on multi-variable polynomial ring, but in general the remainder is not unique.

Theorem 2.21 *Fix a monomial order $\succ$ and a Gröbner basis $\mathcal{G} = \{g_1, \ldots, g_s\}$ for I with respect to $\succ$. Then every $f \in k[x_1, \ldots, x_n]$ can be written as*

$$f = a_1 g_1 + \cdots + a_s g_s + r, \quad a_1, \ldots, a_s, r \in k[x_1, \ldots, x_n] \tag{2.1}$$

where either $r = 0$ or no term of r is divisible by any of $in_\succ(g_1), \ldots, in_\succ(g_s)$. r is unique, and called normal form *of f by $\mathcal{G}$. We write $r = \overline{f}^{\mathcal{G}}$.*

The algorithm below is the algorithm which calculates $a_1, \ldots, a_s, r$ in (2.1), which is called *the division algorithm.*

Algorithm 2.22 (The division algorithm)

Input: f, *Gröbner basis* $\mathcal{G} = \{g_1, \ldots, g_s\}$ *and a term order* $\succ$

Output: $a_1, \ldots, a_s, r$ *for* (2.1)

$a_1 := 0, \ldots, a_s := 0, r := 0$

$p := f$

while $p \neq 0$ **do**

  $i := 1$

  divisionoccured := *false*

  while $i \leq s$ *and divisionoccured* = *false* **do**

    if $in_\succ(g_i)$ *divides* $in_\succ(p)$ **then**

      $a_i := a_i + \dfrac{in_\succ(p)}{in_\succ(g_i)}$

      $p := p - \dfrac{in_\succ(p)}{in_\succ(g_i)} g_i$

      divisionoccured := *true*

    else

      $i := i + 1$

  if *divisionoccured* = *false* **then**

    $r := r + in_\succ(p)$

    $p := p - in_\succ(p)$

Example 2.23 *Let* $\{g_1, g_2\} = \{x + z, y - z\} \subset k[x, y, z]$. $\{g_1, g_2\}$ *is a reduced Gröbner basis for the ideal* $\langle g_1, g_2 \rangle$ *with respect to purely lexicographic order induced by the variable ordering* $x \succ y \succ z$. *Let* $f = xy$ *and divide* f *by* $\{g_1, g_2\}$.

Since $in_\succ(f)$ *is divisible by* $in_\succ(g_1)$, *after the first step of division algorithm* $a_1 = y$ *and* $p = xy - y(x + z) = -yz$. *Next* $in_\succ(p)$ *is divisible by* $in_\succ(g_2)$, *thus* $a_2 = -z$ *and* $p = -z^2$. *Since* $in_\succ(p)$ *is not divisible by neither* $in_\succ(g_1)$ *nor* $in_\succ(g_2)$, *then* $r = z^2$. *As a result, we get the form*

$$xy = y(x + z) - z(y - z) - z^2.$$

■

We give an algorithm to calculate a Gröbner basis by Buchberger [4]. In Buchberger algorithm, *S-polynomial* plays an important role.

Definition 2.24 *Let $f, g \in k[x_1, \dots, x_n]$ be nonzero polynomials.*

(i) *Let $\mathbf{x}^\alpha = in_\succ(f)$ and $\mathbf{x}^\beta = in_\succ(g)$. Let $\gamma = (\gamma_1, \dots, \gamma_n)$, where $\gamma_i = \max(\alpha_i, \beta_i)$. We call $\mathbf{x}^\gamma$ the* least common multiple of *$in_\succ(f)$* and *$in_\succ(g)$, and write $\mathbf{x}^\gamma = LCM(in_\succ(f), in_\succ(g))$.*

(ii) *We define the* S-polynomial of f and g *as*

$$S(f,g) := \frac{\mathbf{x}^\gamma}{in_\succ(f)} \cdot f - \frac{\mathbf{x}^\gamma}{in_\succ(g)} \cdot g,$$

where $\mathbf{x}^\gamma = LCM(in_\succ(f), in_\succ(g))$.

Proposition 2.25 *Let I be an ideal. Then a basis $\mathcal{G} = \{g_1, \dots, g_t\}$ for I is Gröbner basis for I if and only if $\overline{S(g_i, g_j)}^{\mathcal{G}} = 0$ for all pairs $i \neq j$.*

Using this proposition, Buchberger [4] constructed an algorithm which calculates a Gröbner basis.

Algorithm 2.26 (Buchberger Algorithm)
Input: $F = \{f_1, \dots, f_s\} \subset k[x_1, \dots, x_n]$ *and a term order* $\succ$
Output: *Gröbner basis $\mathcal{G}$ for $I = \langle f_1, \dots, f_s \rangle$ with respect to* $\succ$
$\mathcal{G} := F$
repeat
  $\mathcal{G}' := \mathcal{G}$
  for *each pair* $\{p, q\}$, $p \neq q$ *in* $\mathcal{G}'$ **do**
    $S := \overline{S(p,q)}^{\mathcal{G}'}$
    if $S \neq 0$ **then** $\mathcal{G} := \mathcal{G} \cup \{S\}$
until $\mathcal{G} = \mathcal{G}'$

Proposition 2.27 *For any term order and any ideal, a Gröbner basis can be constructed in finite steps by the above algorithm.*

2.2 State Polytopes

In this section, we introduce the *state polytope* [3] of an ideal I. It has the property that the vertices are in a bijection with the distinct reduced Gröbner bases for I.

At first, we review some basic concepts from polyhedral geometry.

Definition 2.28 *A* polyhedron *is an intersection of finitely many closed half-spaces in* $\mathbb{R}^n$.

Definition 2.29 *Let* P *be a polyhedron in* $\mathbb{R}^n$ *and* $c \in \mathbb{R}^n$. *We define the* face *of* P *with respect to* c *by*

$$face_c(P) := \{\mathbf{u} \in P \colon c \cdot \mathbf{u} \geq c \cdot \mathbf{v} \text{ for } {}^\forall \mathbf{v} \in P\}.$$

Example 2.30 *Let*

$$P = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 \colon -2 \leq x_1 \leq 2,\ -2 \leq x_2 \leq 2\}.$$

When $c_1 = (1,1)$, *then* $face_{c_1}(P) = \{(1,1)\}$. *When* $c_2 = (1,0)$, *then* $face_{c_2}(P) = \{(1, x_2) \colon -1 \leq x_2 \leq 1\}$. *(Figure 2.2)* ■

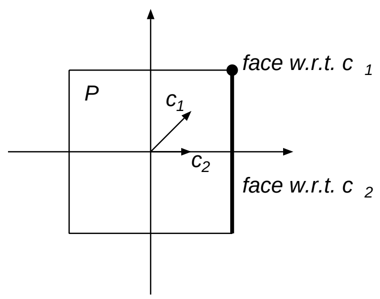

Figure 2.2: P in Example 2.30

Definition 2.31 *Let* $P \subset \mathbb{R}^n$ *be a polyhedron and* F *be a face of* P. *The* normal cone *of* F *at* P *is*

$$\mathcal{N}_P(F) := \{c \in \mathbb{R}^n \colon face_c(P) = F\}.$$

The collection of normal cones as F *ranges over all the faces of* P *is called the* normal fan *of* P *and written by* $\mathcal{N}(P)$.

Example 2.32 *Let* P *be the same polyhedron as Example 2.30. Then the normal cone of* $F_1 = \{(1,1)\}$ *at* P *is the shaded area in Figure 2.3(left). The normal fan of* P *is drawn in Figure 2.3(right). Let*

$$F_1 = \{(1,1)\},\ F_2 = \{(-1,1)\},\ F_3 = \{(-1,-1)\},\ F_4 = \{(1,-1)\}$$

be faces of P. *Then in Figure 2.3(right), the cone* C_i $(i = 1,2,3,4)$ *is the normal cone of* F_i *at* P. ■

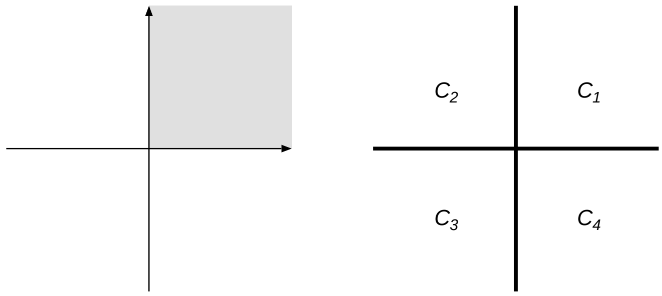

Figure 2.3: The normal cone of F_1 at P (left) and the normal fan of P (right) in Example 2.32.

Next we introduce the *Gröbner fan* [14] of an ideal I and *state polytope* of I. These connect the ideal theory with polyhedral geometry.

Definition 2.33 *Fix $\omega \in \mathbb{R}^n$. For any polynomial $f = \sum c_i \mathbf{x}^{\mathbf{a}_i}$, we define the* initial form *$in_\omega(f)$ to be the sum of all terms $c_i \mathbf{x}^{\mathbf{a}_i}$ such that the inner product $\omega \cdot \mathbf{a}_i$ is maximal. We define the* initial ideal *of I with respect to ω as*

$$in_\omega(I) := \langle in_\omega(f) \colon f \in I \rangle.$$

Remark 2.34 ([20, Proposition 1.11.]) *For any term order $\succ$ and any ideal I, there exists a non-negtive integer vector $\omega \in \mathbb{N}^n$ such that $in_\omega(I) = in_{\succ_\omega(I)}$.*

Definition 2.35 *Let $I \subset k[x_1, \dots, x_n]$ be an ideal. Two weight vectors $\omega_1, \omega_2 \in \mathbb{R}^n$ are called* equivalent *with respect to I if and only if $in_{\omega_1}(I) = in_{\omega_2}(I)$.*

Proposition 2.36 *The set of all weight vectors that are equivalent to $\omega \in \mathbb{R}^n$ form a relatively open polyhedral cone in $\mathbb{R}^n$, the closure of which is called the* Gröbner cone *of ω.*

Definition 2.37 *We define the* Gröbner fan of I *$GF(I)$ to be the collection of all Gröbner cones of I.*

Example 2.38 *Let $I = \langle xy + x + y \rangle \subset k[x, y]$. Then I has three Gröbner fans (Figure 2.4):*

$$C_1 = \{(\omega_1, \omega_2) \colon in_{(\omega_1,\omega_2)}(I) = \langle xy \rangle\}$$
$$C_2 = \{(\omega_1, \omega_2) \colon in_{(\omega_1,\omega_2)}(I) = \langle y \rangle\}$$
$$C_3 = \{(\omega_1, \omega_2) \colon in_{(\omega_1,\omega_2)}(I) = \langle x \rangle\}$$

■

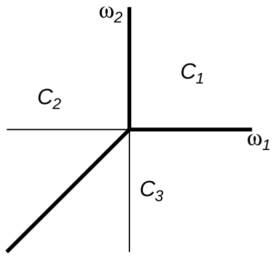

Figure 2.4: The Gröbner fan $GF(I)$ in Example 2.38.

Theorem 2.39 *Let I be a homogeneous ideal in $k[x_1, \ldots, x_n]$. Then there exists a polytope $St(I) \subset \mathbb{R}^n$ whose normal fan $\mathcal{N}(St(I))$ coincides with the Gröbner fan $GF(I)$. This $St(I)$ is called* state polytope *of I.*

Corollary 2.40 *Let I be an ideal in $k[x_1, \ldots, x_n]$. Then I has only finitely many distinct reduced Gröbner bases.*

2.3 Toric Ideals

In this section, we consider $A \in \mathbb{Z}^{d \times n}$ as a set of column vectors $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$. Each vector $\mathbf{a}_i$ is identified with a monomial $\mathbf{t}^{\mathbf{a}_i}$ in the Laurent polynomial ring $k[\mathbf{t}^{\pm 1}] := k[t_1, \ldots, t_d, t_1^{-1}, \ldots, t_d^{-1}]$.

Definition 2.41 *Consider the homomorphism*

$$\pi\colon k[x_1, \ldots, x_n] \longrightarrow k[\mathbf{t}^{\pm 1}], \quad x_i \longmapsto \mathbf{t}^{\mathbf{a}_i}$$

The kernel of π is denoted I_A and called the toric ideal of *A.*

Every vector $\mathbf{u} \in \mathbb{Z}^n$ can be written uniquely as $\mathbf{u} = \mathbf{u}^+ - \mathbf{u}^-$ where $\mathbf{u}^+$ and $\mathbf{u}^-$ are non-negative and have disjoint support. (*Support* is a set of indices of non-zero elements.)

Lemma 2.42

$$I_A = \langle \mathbf{x}^{\mathbf{u}_i^+} - \mathbf{x}^{\mathbf{u}_i^-} : \mathbf{u} \in \ker(A) \cap \mathbb{Z}^n,\ i = 1, \ldots, s \rangle.$$

Furthermore, toric ideal is generated by finite binomials. (A binomial *is a polynomial which consists of two monomials.)*

A Gröbner basis for a toric ideal can be computed as follows:

Algorithm 2.43

Input: *$A \in \mathbb{Z}^{d\times n}$ and a term order $\succ$*

Output: *Gröbner basis for toric ideal I_A with respect to $\succ$*

1. *Introduce $d+n+1$ indeterminates $t_0, t_1, \dots, t_d, x_1, \dots, x_n$. Let $\succ'$ be any elimination order with $\{t_0, \dots, t_d\} \succ' \{x_1, \dots, x_n\}$ whose restriction to $k[x_1, \dots, x_n]$ induces the same total order as $\succ$.*

2. *Compute a reduced Gröbner basis $\mathcal{G}$ for the ideal*

$$\langle t_0 t_1 \cdots t_d - 1, x_1 \mathbf{t}^{\mathbf{a}_1^-} - \mathbf{t}^{\mathbf{a}_1^+}, \dots, x_n \mathbf{t}^{\mathbf{a}_n^-} - \mathbf{t}^{\mathbf{a}_n^+} \rangle.$$

3. *Output the set $\mathcal{G} \cap k[x_1, \dots, x_n]$. This set is a reduced Gröbner basis for I_A with respect to $\succ$.*

Remark 2.44 (See [23].) *If all entries of the matrix A are non-negative, we do not need the variable t_0 and the binomial $t_0 t_1 \cdots t_d - 1$ in the above algorithm.*

We next define two subset of I_A.

Definition 2.45 *A binomial $\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} \in I_A$ is called* circuit *if $\mathbf{u}$ satisfies the following:*

(i) *The support of $\mathbf{u}$ is minimal with respect to inclusion in* $\ker(A)$*.*

(ii) *The coordinates of $\mathbf{u}$ are relatively prime.*

We denote the set of circuits in I_A by $\mathcal{C}_A$.

Definition 2.46 *A binomial $\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} \in I_A$ is called* primitive *if there exists no other binomial $\mathbf{x}^{\mathbf{v}^+} - \mathbf{x}^{\mathbf{v}^-} \in I_A$ such that $\mathbf{x}^{\mathbf{v}^+}$ divides $\mathbf{x}^{\mathbf{u}^+}$ and $\mathbf{x}^{\mathbf{v}^-}$ divides $\mathbf{x}^{\mathbf{u}^-}$. The set of all primitive binomials in I_A is called the* Graver basis of A *and written by Gr_A.*

Let $\mathcal{U}_A$ be the universal Gröbner basis of I_A. Three set $\mathcal{C}_A$, $\mathcal{U}_A$ and Gr_A have the following relation.

Proposition 2.47 *$\mathcal{C}_A \subseteq \mathcal{U}_A \subseteq Gr_A$. If A is a unimodular matrix, then $\mathcal{C}_A = \mathcal{U}_A = Gr_A$.*

Sturmfels [19] showed the single-exponential degree bound for Gröbner bases of toric ideals.

Theorem 2.48 ([19, Theorem 2.3]) *The total degree of a polynomial in any reduced Gröbner basis of I_A is at most $n(n-d)A^d$, where A is the maximum of the Euclidean norms $|\mathbf{a}_1|, \ldots, |\mathbf{a}_n|$.*

Chapter 3

Gröbner Bases for Toric Ideals of Acyclic Tournament Graphs

In the case of toric ideals of the vertex-edge incidence matrices of acyclic tournament graphs, the elements in universal Gröbner bases correspond to the circuits of the graphs. Thus for some term orders, the reduced Gröbner bases for toric ideals are characterized in terms of circuits.

In this chapter, we generate reduced Gröbner bases for toric ideals of acyclic tournament graphs with respect to some specific term orders.

3.1 Toric Ideals of Acyclic Tournament Graphs

In this section, we define toric ideals of acyclic tournament graphs, and show that the elements in universal Gröbner bases correspond to the circuits of graphs. We also define two positive gradings, one is that the degree of each variable is 1, and the other is that toric ideal of acyclic tournament graph becomes homogeneous.

Let D_n be an acyclic tournament graph with n vertices $\{1, 2, \ldots, n\}$ whose edge (i, j) $(i < j)$ is directed from i to j, and $m = \binom{n}{2}$ be the number of edges in D_n. Let A_n be the vertex-edge incidence matrix of D_n. We associate each edge (i, j) with a variable x_{ij}, and we consider the polynomial ring $k[x_{ij} \colon 1 \leq i < j \leq n]$. We define the *toric ideal* of D_n as

$$I_{A_n} := \langle \mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} : \mathbf{u} \in \ker(A_n) \cap \mathbb{Z}^m \rangle.$$

Thus the elements in I_{A_n} correspond to the disjoint sum of cycles of D_n.

We associate the circuits of I_{A_n} with the circuits of D_n.

Remark 3.1 *In this thesis, we define* a circuit of D_n *as a simple cycle.*

Definition 3.2 *Let C be a circuit of D_n. If we fix a direction of C, we can partition C into two sets of edges C^+ and C^- such that C^+ is the set of forward edges and C^- is the set of backward edges. Then the vector $\mathbf{x} = (x_{12}, x_{13}, \dots, x_{n-1,n}) \in \mathbb{R}^m$ defined by*

$$x_{ij} = \begin{cases} 1 & \text{if } (i,j) \in C^+ \\ -1 & \text{if } (i,j) \in C^- \\ 0 & \text{if } (i,j) \notin C \end{cases} \qquad (1 \le i < j \le n)$$

is called the incidence vector *of C.*

Lemma 3.3 ([2, Proposition 2.17]) *A binomial $\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} \in I_{A_n}$ is the circuit of I_{A_n} if and only if $\mathbf{u}$ is the incidence vector of the circuit of D_n.*

For the case of I_{A_n}, since A_n is unimodular, all inclusions in Proposition 2.47 are equals.

Proposition 3.4 *For the case of I_{A_n}, $\mathcal{C}_{A_n} = \mathcal{U}_{A_n} = Gr_{A_n}$.*

Proof: If $\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} \in Gr_{A_n}$ is not the circuit of I_{A_n}, then there exists the circuit $\mathbf{x}^{\mathbf{c}^+} - \mathbf{x}^{\mathbf{c}^-} \in I_{A_n}$ such that

$$supp(\mathbf{c}^+) \subseteq supp(\mathbf{u}^+), \quad supp(\mathbf{c}^-) \subseteq supp(\mathbf{u}^-).$$

By Lemma 3.3, since all elements of $\mathbf{c}^+$ and $\mathbf{c}^-$ are either 0 or 1, $\mathbf{x}^{\mathbf{u}^+}$ is divisible by $\mathbf{x}^{\mathbf{c}^+}$ and $\mathbf{x}^{\mathbf{u}^-}$ is divisible by $\mathbf{x}^{\mathbf{c}^-}$. Then $\mathbf{u}$ is not primitive. ■

Corollary 3.5 *The universal Gröbner basis $\mathcal{U}_{A_n}$ is the set of binomials which correspond to the circuits of D_n.*

Corollary 3.6 *The number of elements in $\mathcal{U}_{A_n}$ equals the number of circuits, i.e.*

$$\binom{n}{3} + \frac{3!}{2}\binom{n}{4} + \dots + \frac{(n-1)!}{2}\binom{n}{n} \ge \binom{n}{3} + \binom{n}{4} + \dots + \binom{n}{n} = 2^n - \frac{n^2+n+2}{2}.$$

In particular, the number of elements in $\mathcal{U}_{A_n}$ is of exponential order with respect to n.

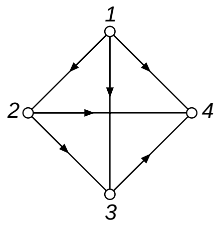

Figure 3.1: D_4

Example 3.7 *Let $n = 4$. The vertex-edge incidence matrix A_4 is*

$$A_4 = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{pmatrix}$$

The circuit $C = 1, 2, 3, 1$ in D_4 corresponds to the vector $(1, -1, 0, 1, 0, 0) \in \ker(A)$ and the circuit $x_{12}x_{23} - x_{13} \in I_{A_4}$. ■

Example 3.8 *We calculate the Gröbner bases of D_4. If $\succ_1$ is the purely lexicographic order induced by the variable ordering $x_{12} \succ x_{13} \succ x_{14} \succ x_{23} \succ x_{24} \succ x_{34}$, then reduced Gröbner basis with respect to $\succ_1$ is*

$$\{\underline{x_{12}x_{23}} - x_{13}, \underline{x_{12}x_{24}} - x_{14}, \underline{x_{13}x_{24}} - x_{14}x_{23}, \underline{x_{13}x_{34}} - x_{14}, \underline{x_{23}x_{34}} - x_{24}\}.$$

If $\succ_2$ is the purely lexicographic order induced by the variable ordering $x_{13} \succ x_{24} \succ x_{23} \succ x_{34} \succ x_{12} \succ x_{14}$, then reduced Gröbner basis with respect to $\succ_2$ is

$$\{\underline{x_{12}x_{23}x_{34}} - x_{14}, \underline{x_{24}} - x_{23}x_{34}, \underline{x_{13}} - x_{12}x_{23}\}.$$

The universal Gröbner basis $\mathcal{U}_{A_4}$ is

$$\{x_{12}x_{23}x_{34} - x_{14}, x_{12}x_{23} - x_{13}, x_{12}x_{24} - x_{13}x_{34}, x_{12}x_{24} - x_{14},$$

$$x_{13}x_{24} - x_{14}x_{23}, x_{13}x_{34} - x_{14}, x_{23}x_{34} - x_{24}\},$$

which corresponds the set of circuits in D_4. ■

As in the above example, the number and degree of elements in reduced Gröbner bases vary if the term order changes.

I_{A_n} is not homogeneous with respect to a positive grading $\deg(x_{12}) = \cdots = \deg(x_{n-1,n}) = 1$. But we can change the positive grading such that I_{A_n} becomes homogeneous.

Theorem 3.9 *If we set a positive grading as*

$$\deg(x_{ij}) = j - i, \quad 1 \leq i < j \leq n, \tag{3.1}$$

then I_{A_n} is a homogeneous ideal.

Proof: It suffices to show that any elements in $\mathcal{U}_{A_n}$ are homogeneous with respect to the positive grading (3.1).

Let $C = i_1, i_2, \ldots, i_s, i_1$ be a circuit in D_n. Let $C^+ := \{k \colon i_k < i_{k+1}\}$ and $C^- := \{k \colon i_k > i_{k+1}\}$ (we set $i_{s+1} = i_1$). The binomial f_C corresponding to C is

$$f_C = \prod_{k \in C^+} x_{i_k i_{k+1}} - \prod_{k \in C^-} x_{i_{k+1} i_k}.$$

Then, since $C^+ \cap C^- = \emptyset$,

$$\begin{aligned} \deg\left(\prod_{k \in C^+} x_{i_k i_{k+1}}\right) - \deg\left(\prod_{k \in C^-} x_{i_k i_{k+1}}\right) &= \sum_{k \in C^+} (i_{k+1} - i_k) - \sum_{k \in C^-} (i_k - i_{k+1}) \\ &= \sum_{k=1}^{s} (i_{k+1} - i_k) \\ &= 0. \end{aligned}$$

Thus f_C is homogeneous. ■

In the rest of this chapter, we consider two positive gradings:

1. $\deg(x_{ij}) = 1$ for any $1 \leq i < j \leq n$
2. $\deg(x_{ij}) = j - i$ for any $1 \leq i < j \leq n$

We call the former *standard grading* and the latter *graphical grading*.

Remark 3.10 *Toric ideals of acyclic tournament graphs are homogeneous with respect to graphical grading. Thus state polytopes can be defined, and all reduced Gröbner bases can be enumerated using TiGERS [11].* (*See Chapter 4*)

3.2 Some Reduced Gröbner Bases of I_{A_n}

In this section, with respect to two positive gradings in Section 3.1, we show that the elements of reduced Gröbner bases with respect to some specific term orders can be given in terms of graphs. As a corollary, we can show that there exist term orders for which reduced Gröbner bases remain in polynomial order.

3.2.1 Case of Graphical Grading

We consider the graphical positive grading. We first show the term order for which the elements in reduced Gröbner basis correspond to the circuits of length three and some circuits of length four of D_n.

Theorem 3.11 *Let $\succ_1$ be the purely lexicographic order induced by the following variable ordering:*

$$x_{ij} \succ x_{kl} \Longleftrightarrow i < k \text{ or } (i = k \text{ and } j < l).$$

Let

$$g_{ijk} := \underline{x_{ij}x_{jk}} - x_{ik} \quad (1 \leq i < j < k \leq n),$$

$$g_{ijkl} := \underline{x_{ik}x_{jl}} - x_{il}x_{jk} \quad (1 \leq i < j < k < l \leq n).$$

Then reduced Gröbner basis $\mathcal{G}_1$ of I_{A_n} with respect to $\succ_1$ is

$$\mathcal{G}_1 = \{g_{ijk} \colon 1 \leq i < j < k \leq n\} \cup \{g_{ijkl} \colon 1 \leq i < j < k < l \leq n\}.$$

In particular, the number of elements in $\mathcal{G}_1$ equals $\binom{n}{3} + \binom{n}{4}$ and the degree equals $2n - 4$.

g_{ijk} corresponds to the circuit i, j, k, i of D_n, and g_{ijkl} corresponds to the circuit i, k, j, l, i of D_n. Thus the set $\{g_{ijk} \colon 1 \leq i < j < k \leq n\}$ corresponds to all of the circuits of length three, and $\{g_{ijkl} \colon 1 \leq i < j < k < l\}$ corresponds to some of circuits of length four (Figure 3.2).

Proof: For any circuit of length three defined by three vertices i, j, k $(i < j < k)$, the associated binomial equals $\underline{x_{ij}x_{jk}} - x_{ik}$, which is g_{ijk}.

The circuits defined by four vertices $i < j < k < l$ are

$$C_1 := i, j, k, l, i, \quad C_2 := i, j, l, k, i, \quad C_3 := i, k, j, l, i$$

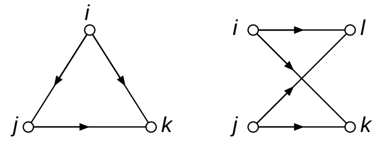

Figure 3.2: The circuit corresponding to g_{ijk} and the circuit corresponding to g_{ijkl}.

and their opposites(Figure 3.3). The binomial which corresponds to C_1 or its opposite is $\underline{x_{ij}x_{jk}x_{kl}} - x_{il}$, whose initial term is divisible by $in_{\succ_1}(g_{ijk}) = x_{ij}x_{jk}$. Similarly, the initial term of binomial which corresponds to C_2 or its opposite is divisible by $in_{\succ_1}(g_{ijl})$. The binomial which corresponds to C_3 or its opposite is $\underline{x_{ik}x_{jl}} - x_{il}x_{jk}$, which is g_{ijkl}.

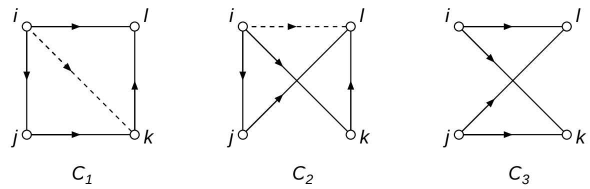

Figure 3.3: The circuits C_1, C_2, C_3.

Let C be a circuit of length more than 5. Let i_1 be the vertex whose label is minimum in C, and $C := i_1, i_2, \dots, i_s, i_1$. Without loss of generality, we set $i_2 < i_s$. Let f_C be the binomial corresponding to C, then $in_{\succ_1}(f_C)$ is the product of all variables whose associated edges have same direction with (i_1, i_2) on C. We show that $in_{\succ_1}(f_C)$ is divisible by the initial term of a binomial in $\mathcal{G}_1$, which implies that $\mathcal{G}_1$ is Gröbner basis of I_{A_n} with respect to $\succ_1$.

If $i_2 < i_3$, then (i_1, i_2) and (i_2, i_3) have same direction on C. Thus the variables $x_{i_1i_2}$ and $x_{i_2i_3}$ appear in $in_{\succ_1}(f_C)$, and $in_{\succ_1}(f_C)$ is divisible by $in_{\succ_1}(g_{i_1i_2i_3})$ (Figure 3.4).

If $i_2 > i_3$, since $i_3 < i_2 < i_s$, there exists some k $(3 \leq k < s)$ such that $i_1 < i_k < i_2 < i_{k+1}$. Then the variables $x_{i_1i_2}$ and $x_{i_ki_{k+1}}$ appear in $in_{\succ_1}(f_C)$, and $in_{\succ_1}(f_C)$ is divisible by $in_{\succ_1}(g_{i_1i_ki_2i_{k+1}})$ (Figure 3.5).

Any term of g_{ijk} is not divisible by the initial term of any other binomial in $\mathcal{G}_1$, and so as g_{ijkl}. This implies that $\mathcal{G}_1$ is reduced.

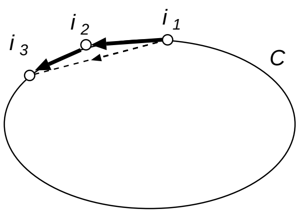

Figure 3.4: $x_{i_1 i_2}$ and $x_{i_2 i_3}$ appear in $in_{\succ_1}(f_C)$.

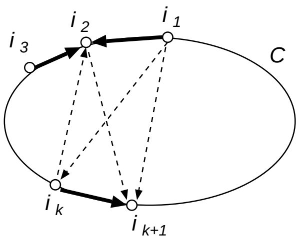

Figure 3.5: $x_{i_1 i_2}$ and $x_{i_k i_{k+1}}$ appear in $in_{\succ_1}(f_C)$.

The degree of g_{ijk} equals $k-i$, and that of g_{ijkl} equals $(k-i)+(l-j) = k+l-i-j$. Thus the degree of $\mathcal{G}_1$ equals $n+(n-1)-1-2 = 2n-4$. ■

Example 3.12 *Let $n = 5$. Then $\succ_1$ is the purely lexicographic order induced by the variable ordering*

$$x_{12} \succ x_{13} \succ x_{14} \succ x_{15} \succ x_{23} \succ x_{24} \succ x_{25} \succ x_{34} \succ x_{35} \succ x_{45}.$$

The reduced Gröbner basis of I_{A_5} with respect to $\succ_1$ consists of 15 binomials:

$$\underline{x_{12}x_{23}} - x_{13}, \underline{x_{12}x_{24}} - x_{14}, \underline{x_{12}x_{25}} - x_{15}, \underline{x_{13}x_{34}} - x_{14}, \underline{x_{13}x_{35}} - x_{15}$$

$$\underline{x_{14}x_{45}} - x_{15}, \underline{x_{23}x_{34}} - x_{24}, \underline{x_{23}x_{35}} - x_{25}, \underline{x_{24}x_{45}} - x_{25}, \underline{x_{34}x_{45}} - x_{35}$$

$$\underline{x_{13}x_{24}} - x_{14}x_{23}, \underline{x_{13}x_{25}} - x_{15}x_{23}, \underline{x_{14}x_{25}} - x_{15}x_{24}, \underline{x_{14}x_{35}} - x_{15}x_{34}, \underline{x_{24}x_{35}} - x_{25}x_{34}.$$

■

Next we show the term order for which the elements in reduced Gröbner basis correspond to the fundamental circuits for a certain spanning tree of D_n.

Theorem 3.13 *Let $\succ_2$ be the purely lexicographic order induced by the following variable ordering:*

$$x_{ij} \succ x_{kl} \Longleftrightarrow i < k \text{ or } (i = k \text{ and } j > l).$$

Let

$$g_{ij} := \underline{x_{ij}} - x_{i,i+1}x_{i+1,i+2}\cdots x_{j-1,j} \quad (1 \leq i < j-1 < n)$$

Then reduced Gröbner basis $\mathcal{G}_2$ of I_{A_n} with respect to $\succ_2$ is

$$\mathcal{G}_2 = \{g_{ij} : 1 \leq i < j-1 < n\}.$$

In particular, the number of elements in $\mathcal{G}_2$ *equals* $\binom{n}{2} - (n-1)$ *and the degree equals* $n-1$.

The elements of reduced Gröbner basis $\mathcal{G}_2$ correspond to the set of fundamental circuits of D_n for the spanning tree

$$T := \{(i, i+1) \colon 1 \leq i < n\}.$$

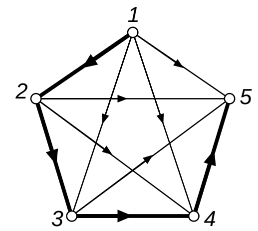

Figure 3.6: The spanning tree T in D_5

Proof: Let C be a circuit which is not the fundamental circuit of T. Let i_1 be the vertex whose label is minimum in C, and $C := i_1, i_2, \ldots, i_s, i_1$. Without loss of generality, we set $i_2 < i_s$. Then the variable $x_{i_1 i_s}$ appears in the initial term of associated binomial f_C (Figure 3.7). Thus $in_{\succ_2}(f_C)$ is divisible by $in_{\succ_2}(g_{i_1 i_s})$. It implies that $\mathcal{G}_2$ is Gröbner basis of I_{A_n} with respect to $\succ_2$.

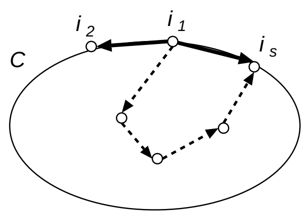

Figure 3.7: $in_{\succ_2}(f_C)$ is divisible by the initial term of the binomial which corresponds to the fundamental circuit of (i_1, i_s).

The initial term of g_{ij} corresponds to an edge which is not contained in T, and other term corresponds to several edges which are contained in T. Thus any term of g_{ij} is not divisible by the initial term of any other binomial in $\mathcal{G}_2$, which implies that $\mathcal{G}_2$ is reduced.

The degree of g_{ij} equals $j - i$. Thus the degree of $\mathcal{G}_2$ equals $n-1$. ■

Example 3.14 *Let $n = 5$. Then $\succ_2$ is the purely lexicographic order induced by the variable ordering*

$$x_{15} \succ x_{14} \succ x_{13} \succ x_{12} \succ x_{25} \succ x_{24} \succ x_{23} \succ x_{35} \succ x_{34} \succ x_{45}.$$

The reduced Gröbner basis of I_{A_5} with respect to $\succ_2$ consists of 6 binomials:

$$\underline{x_{13}} - x_{12}x_{23}, \underline{x_{14}} - x_{12}x_{23}x_{34}, \underline{x_{15}} - x_{12}x_{23}x_{34}x_{45}$$

$$\underline{x_{24}} - x_{23}x_{34}, \underline{x_{25}} - x_{23}x_{34}x_{45}, \underline{x_{35}} - x_{34}x_{45}.$$

■

As we show in next chapter, this is the case that the number of elements in reduced Gröbner basis is minimum for any term order.

We last show that there exist two term orders for which reduced Gröbner bases are same as $\mathcal{G}_1$.

Theorem 3.15 *Let $\succ_3$ be the purely lexicographic order induced by the following variable ordering:*

$$x_{ij} \succ x_{kl} \Longleftrightarrow j < l \text{ or } (j = l \text{ and } i < k).$$

Then reduced Gröbner basis of I_{A_n} with respect to $\succ_3$ is same as $\mathcal{G}_1$ in Theorem 3.11.

Proof: For the circuits of length less than four, we can show similarly as the proof of Theorem 3.11.

Let C be a circuit of length more than five. Let i_1 be the vertex whose label is minimum in C, and $C := i_1, i_2, \ldots, i_s, i_1$. Without loss of generality, we set $i_2 < i_s$. Let f_C be the associated binomial.

Let T_C be a subset of vertices in C such that

$$T_C := \{i_s \in C \colon i_{s-1} < i_s\} \cup \{i_s \in C \colon i_{s+1} < i_s\}.$$

(We set $i_{s+1} = i_1$) This is the set of vertices which are the terminal points of edges in C. Let i_k be the vertex whose label is minimum in T_C.

If $k = 2$, then the variable $x_{i_1 i_2}$ is the maximum variable in f_C with respect to $\succ_3$. Then $in_{\succ_3}(f_C)$ is the product of all variables whose associated edges have same direction with (i_1, i_2) on C. In this case, we can show that $\mathcal{G}_1$ is the reduced Gröbner basis with respect to $\succ_3$ by similar way as Theorem 3.11.

Let $k \neq 2$. If $i_{k-1} < i_k < i_{k+1}$ (Figure 3.8), the variable $x_{i_{k-1}i_k}$ is the maximum variable in f_C by the choice of k. Then the variables $x_{i_{k-1}i_k}$ and $x_{i_k i_{k+1}}$ appear in $in_{\succ_3}(f_C)$, and $in_{\succ_3}(f_C)$ is divisible by $in_{\succ_3}(g_{i_{k-1}i_k i_{k+1}})$. Similarly we can show for the case of $i_{k-1} > i_k > i_{k-1}$.

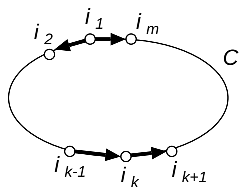

Figure 3.8: If $i_{k-1} < i_k < i_{k+1}$, $in_{\succ_3}(f_C)$ is divisible by $in_{\succ_3}(g_{i_{k-1}i_k i_{k+1}})$.

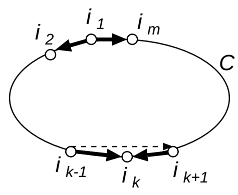

Figure 3.9: The case $i_{k-1} < i_{k+1} < i_k$.

Let $i_{k-1} < i_k$ and $i_{k+1} < i_k$ (Figure 3.9). If $i_{k-1} < i_{k+1}$, then the variable $x_{i_{k-1}i_k}$ is the maximum variable in f_C. Thus the variable $x_{i_{k-1}i_k}$ appears in $in_{\succ_3}(f_C)$. By the choice of k, it can be shown that $i_{k-1} < i_{k+1} < i_k < i_{k+2}$. (We set $i_{m+2} = i_2$.) In fact, if $i_{k+2} < i_{k+1}$ (Figure 3.10 left), then $i_{k+2} < i_{k+1} < i_k$. Thus i_{k+1} is the vertex whose label is minimum in T_C, which implies i_{k+1} contradicts the choice of k. If $i_{k+1} < i_{k+2} < i_k$ (Figure 3.10 right), then i_{k+2} contradicts the choice of k.

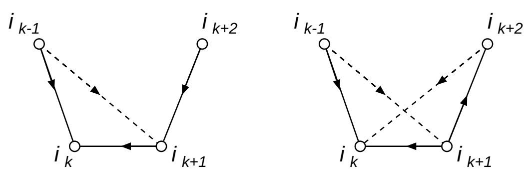

Figure 3.10: i_{k+1} (left) or i_{k+2} (right) contradict the choice of k.

Since $i_{k-1} < i_{k+1} < i_k < i_{k+2}$, the variables $x_{i_{k-1}i_k}$ and $x_{i_{k+1}i_{k+2}}$ appear in $in_{\succ_3}(f_C)$. Thus $in_{\succ_3}(f_C)$ is divisible by $in_{\succ_3}(g_{i_{k-1}i_{k+1}i_k i_{k+2}})$. If $i_{k-1} > i_{k+1}$, similarly we can show that $in(f_C)$ is divisible by $in_{\succ_3}(g_{i_{k+1}i_{k-1}i_k i_{k-2}})$. Thus $\mathcal{G}_1$ is the Gröbner basis of I_{A_n} with respect to $\succ_3$.

The proof that $\mathcal{G}_1$ is reduced is same as the proof of Theorem 3.11. ■

Example 3.16 *Let* $n = 5$. *Then* $\succ_3$ *is the purely lexicographic order induced by the variable ordering*

$$x_{12} \succ x_{13} \succ x_{23} \succ x_{14} \succ x_{24} \succ x_{34} \succ x_{15} \succ x_{25} \succ x_{35} \succ x_{45}.$$

The reduced Gröbner basis of I_{A_5} *with respect to* $\succ_3$ *is same as Example 3.12.* ■

We consider the degree lexicographic order. In graphical grading case, since any elements f_C which corresponding to the circuit C of D_n are homogeneous, $in(f_C)$ with respect to the degree lexicographic order equal $in(f_C)$ with respect to the purely lexicographic order induced by same variable ordering. Thus we can show the degree lexicographic versions of Theorem 3.11, 3.13 and 3.15 by the same proofs for these theorems.

Corollary 3.17 *Let* $\succ_4$ *be the degree lexicographic order induced by the following variable ordering:*

$$x_{ij} \succ x_{kl} \Longleftrightarrow i < k \text{ or } (i = k \text{ and } j < l).$$

Then reduced Gröbner basis of I_{A_n} *with respect to* $\succ_4$ *is same as* $\mathcal{G}_1$ *in Theorem 3.11.*

Corollary 3.18 *Let* $\succ_5$ *be the degree lexicographic order induced by the following variable ordering:*

$$x_{ij} \succ x_{kl} \Longleftrightarrow i < k \text{ or } (i = k \text{ and } j > l).$$

Then reduced Gröbner basis of I_{A_n} *with respect to* $\succ_5$ *is same as* $\mathcal{G}_2$ *in Theorem 3.13.*

Corollary 3.19 *Let* $\succ_6$ *be the degree lexicographic order induced by the following variable ordering:*

$$x_{ij} \succ x_{kl} \Longleftrightarrow j < l \text{ or } (j = l \text{ and } i < k).$$

Then reduced Gröbner basis of I_{A_n} *with respect to* $\succ_6$ *is same as* $\mathcal{G}_1$ *in Theorem 3.11.*

3.2.2 Case of Standard Grading

We consider the standard positive grading.

For the purely lexicographic orders, since we decide the initial term of the binomial by only variable ordering (i.e. without comparing the degree), initial term of each binomial is same as that with respect to graphical grading. Thus we can show the standard grading versions of Theorem 3.11, 3.13 and 3.15 by the same proofs for these theorems. In this case, only the degree changes.

Theorem 3.20 *Let $\succ_1'$ be the purely lexicographic order induced by the following variable ordering:*

$$x_{ij} \succ x_{kl} \Longleftrightarrow i < k \text{ or } (i = k \text{ and } j < l).$$

Then reduced Gröbner basis of I_{A_n} with respect to $\succ_1'$ is same as $\mathcal{G}_1$ in Theorem 3.11. In particular, the degree of $\mathcal{G}_1$ equals 2.

Theorem 3.21 *Let $\succ_2'$ be the purely lexicographic order induced by the following variable ordering:*

$$x_{ij} \succ x_{kl} \Longleftrightarrow i < k \text{ or } (i = k \text{ and } j > l).$$

Then reduced Gröbner basis of I_{A_n} with respect to $\succ_2'$ is same as $\mathcal{G}_2$ in Theorem 3.13. In particular, the degree of $\mathcal{G}_2$ equals $n-1$.

Theorem 3.22 *Let $\succ_3'$ be the purely lexicographic order induced by the following variable ordering:*

$$x_{ij} \succ x_{kl} \Longleftrightarrow j < l \text{ or } (j = l \text{ and } i < k).$$

Then reduced Gröbner basis of I_{A_n} with respect to $\succ_3'$ is same as $\mathcal{G}_1$ in Theorem 3.11.

But in the case of the degree lexicographic order, since initial term may differ from that with respect to purely lexicographic order, we can not extend the above theorems to the degree lexicographic order. For example, in Theorem 3.21, for the purely lexicographic order the initial term of g_{ij} is x_{ij}, but for the degree lexicographic order the initial term of g_{ij} is $x_{i,i+1}x_{i+1,i+2}\cdots x_{j-1,j}$.

Fortunately, we can extend Theorem 3.20 to the degree lexicographic order.

Theorem 3.23 *Let $\succ'_4$ be the degree lexicographic order induced by the following variable ordering:*

$$x_{ij} \succ x_{kl} \Longleftrightarrow i < k \text{ or } (i = k \text{ and } j < l).$$

Then reduced Gröbner basis of I_{A_n} with respect to $\succ'_4$ is same as $\mathcal{G}_1$ in Theorem 3.11.

Proof: For the circuits of length less than four, we can show similarly as the proof of Theorem 3.11.

Let C be a circuit of length more than five. Let i_1 be the vertex whose label is minimum in C, and i_2 be the vertex adjacent to i_1 satisfying the following: let C_1 be the set of edges in C whose direction in C are same as (i_1, i_2) and C_2 be the set of edges in C which do not contained in C_1, then the cardinality of C_1 is more than that of C_2, or if the cardinality equals, then i_2 is the vertex adjacent to i_1 in C whose label is minimum. We write $C := i_1, i_2, \cdots, i_s, i_1$. Let f_C be the associated binomial. Then $in_{\succ'_4}(f_C)$ is product of all variables whose associated edges are contained in C_1.

If there exists k which satisfies $i_{k-1} < i_k < i_{k+1}$, then the variables $x_{i_{k-1}i_k}$ and $x_{i_k i_{k+1}}$ appear in $in_{\succ'_4}(f_C)$. Thus $in_{\succ'_4}(f_C)$ is divisible by $in_{\succ'_4}(g_{i_{k-1}i_k i_{k+1}})$.

If there does not exist such k, then between any two edges which are contained in C_1, there exists at least one edge which are contained in C_2 (Figure 3.11). Then by the choice of i_2, the cardinality of C_1 equals that of C_2. Thus $i_3 < i_2 < i_s$ by hypothesis, and there exists k $(3 \leq k < s)$ such that $i_1 < i_k < i_2 < i_{k+1}$. Then the variables $x_{i_1 i_2}$ and $x_{i_k i_{k+1}}$ appear in $in_{\succ'_4}(f_C)$, and $in_{\succ'_4}(f_C)$ is divisible by $in_{\succ'_4}(g_{i_1 i_k i_2 i_{k+1}})$.

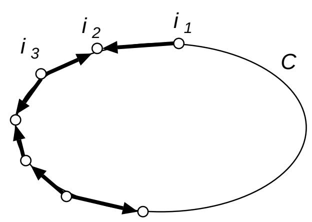

Figure 3.11: Between any two edges which are contained in C_1, there exist at least one edge which is contained in C_2.

The proof that $\mathcal{G}_1$ is reduced is same as the proof of Theorem 3.11. ∎

Chapter 4

Bounds for Size of Gröbner Bases for Various Term Orders

In this chapter, we deal with the number of elements and the degree of reduced Gröbner bases with respect to various term orders. Generally the degree of reduced Gröbner bases for toric ideals is at most of exponential order [19], but the number of elements are not well understood. For the case of acyclic tournament graphs, since those vertex-edge incidence matrices are unimodular, the size and degree of reduced Gröbner bases may be bounded.

4.1 Bound for Degree of Gröbner Bases

As we have shown in Theorem 2.48, the degree of elements in reduced Gröbner bases of general toric ideals are at most exponential order. Since the elements of toric ideals of acyclic tournament graphs correspond to the circuits in the graphs, we can bound the degree of elements in reduced Gröbner bases in both of the cases graphic grading and standard grading.

4.1.1 Case of Graphical Grading

We first consider the case of graphical positive grading.

Theorem 4.1 *The lower bound for the degree of elements in reduced Gröbner bases for I_{A_n} is $n-2$.*

Proof: It suffices to show that any reduced Gröbner bases contain the binomial of degree more than $n-2$.

Because of the definition of Gröbner basis, any reduced Gröbner basis has an element g such that $in(g)$ divides the initial term of the binomial $f := x_{1,n-1}x_{n-1,n} - x_{1n}$ corresponding the cycle $1, n-1, n, 1$.

If $in(f) = x_{1n}$, then $in(g) = x_{1n}$ and the degree of $in(g)$ equals $n-1$. If $in(f) = x_{1,n-1}x_{n-1,n}$, then $in(g)$ contains the variable $x_{1,n-1}$. In fact, if $in(g)$ does not contain $x_{1,n-1}$, then $in(g) = x_{n-1,n}$. But the cycle which passes the edge $(n-1, n)$ always passes at least one of the edge $(i, n-1)$ $(1 \leq i \leq n-2)$, this is contradiction. Thus $\deg(g) \geq n-2$. ■

Theorem 4.2 *The upper bound for the degree of elements in reduced Gröbner bases for I_{A_n} is $O(n^2)$.*

Proof: Because of Corollary 3.5, the elements in reduced Gröbner bases correspond to the circuits in D_n. The length of the circuit in D_n is at most n. But the direction of at least one edge is opposite since the graph is acyclic. Thus each term of elements in reduced Gröbner bases contains at most $n-1$ variables. Since the degree of each variable is less than $n-1$, the degree of any elements is at most $(n-1)^2 = O(n^2)$. ■

4.1.2 Case of Standard Grading

We next consider the case of standard grading case. In this case, the degree becomes linear order.

Theorem 4.3 *The minimum value of degree of elements in reduced Gröbner bases for I_{A_n} is 2. The basis we have shown in Theorem 3.20 is the example achieving this bound.*

Proof: The length of the circuit in D_n is at least 3, but the direction of at least one edge is opposite. Thus the degree of any elements in reduced Gröbner bases is at least 2. ■

Theorem 4.4 *The maximum value of degree of elements in reduced Gröbner bases for I_{A_n} is $n-1$. The basis we have shown in Theorem 3.21 is the example achieving this bound.*

Proof: Because of Corollary 3.5, the elements in reduced Gröbner bases correspond to the circuits in D_n. The length of the circuit in D_n is at most n. But the direction of at least one edge is opposite since the graph is acyclic. Thus the number of edges in circuit whose direction are same is at most $n-1$, which implies the upper bound of the degree is $n-1$. ■

4.2 Bound for Number of Elements in Gröbner Bases

The number of elements in reduced Gröbner bases is not well understood. But since the reduced Gröbner bases for toric ideals of acyclic tournament graphs are the bases for the cycle spaces of the graphs, we can show that the reduced Gröbner basis in the previous chapter is the example achieving minimum number of elements. To analyze the upper bound, we calculate all reduced Gröbner bases using TiGERS [11] for small n and show result.

We also consider the reduced Gröbner bases for toric ideals of acyclic tournament graphs with respect to purely lexicographic orders. We implement the algorithm to check whether the Gröbner basis is the basis with respect to some purely lexicograhpic order [20] and experiment on small n.

4.2.1 Case of General Term Orders

By Remark 2.34, since $in_\omega(I)$ is independent of the grading, the set of all reduced Gröbner bases for any grading equals the set of all reduced Gröbner bases for standard grading. Thus for bounding the number of elements in reduced Gröbner bases, we have only to consider the case of graphical grading.

In the rest of this chapter, we consider the standard grading.

Theorem 4.5 *The minimum number of elements in reduced Gröbner bases for I_{A_n} is $\binom{n}{2} - (n-1)$. The basis we have shown in Theorem 3.13 is the example achieving this bound.*

Proof: Because of Proposition 2.18, the number of elements in reduced Gröbner basis is more than the number of elements in the basis for I_{A_n}. Since I_{A_n} corresponds to the cycle space of D_n, the number of elements in the basis for I_{A_n} equals the dimension of the cycle space of D_n, which is $\binom{n}{2} - (n-1)$. ■

To analyze the upper bound for the number of elements in reduced Gröbner bases, we calculate all reduced Gröbner bases for small n using TiGERS [11]. TiGERS is a software system implemented in C which computes the state polytope of a homogeneous toric ideal [12]. Table 4.1 is the result for $n = 4, 5, 6, 7$. All the experiments were done on Sun UltraSPARC-II, 360 MHz workstation with 1GB memory.

n	# variables	# GB	max. of elements	min. of elements	time
4	6	10	5	3	0.02 s
5	10	211	15	6	0.99 s
6	15	48312	37	10	2 days
7	21	$\geq$ 37665	$\geq$ 75	15	$\geq$ 15 days

Table 4.1: The number of reduced Gröbner bases (#GB), maximum of the number of elements (max. of elements), minimum of the number of elements (min. of elements), and timing.

For $n \leq 5$, the reduced Gröbner basis in Theorem 3.11 is the example achieving maximum elements, but it is not for $n \geq 6$. For $n = 6$, as we show in next section, the Gröbner bases of size 37 are not the bases with respect to purely lexicographic orders. Thus the reduced Gröbner bases which achieve the maximum number of elements seem to be complicated and difficult to characterize.

4.2.2 Case of Lexicographic Orders

We consider the reduced Gröbner bases for I_{A_n} with respect to purely lexicographic orders. The reasons to consider purely lexicographic orders are the following:

- Generally, it is hard to compute Gröbner basis with respect to purely lexicographic order. (Time and space complexity become very large while running the Buchberger algorithm.)
- Generally, the Gröbner bases with respect to purely lexicographic orders are useful to solve the system of polynomial equations by elimination method.

- Using elimination order, Gröbner bases for toric ideals of the subgraphs of D_n can be computed easily.

- Gröbner bases with respect to purely lexicographic order are independent of the positive grading.

We first show an algorithm to check whether Gröbner basis is the basis with respect to some purely lexicographic orders.

Algorithm 4.6 ([20])

Input: *A reduced Gröbner basis* $\mathcal{G} \subset k[x_1, \dots, x_n]$

Output: *"Yes" if* $\mathcal{G}$ *can be Gröbner basis with respect to some purely lexicographic order, "No" if not.*

0. $F := \mathcal{G}$, $X := \{x_1, \dots, x_n\}$

1. *Find a variable* $x_i \in X$ *which appears only in the initial terms of binomials in* F *or do not appear in the binomials in* F*. If there exists no such* $x_i \in X$*, then output "No" and exit.*

2. *Remove all monomials which contains* x_i *from* F*.*

3. $X := X \setminus \{x_i\}$

4. *If* $F = \emptyset$*, then output "Yes" and exit. If* $X \neq \emptyset$*, return* **1**.

Proof of correctness of Algorithm 4.6:

If the output is "Yes", let $x_{i_1}, \dots, x_{i_k}$ be the order of variables which was removed in Step 3. of Algorithm 4.6. Then clearly $\mathcal{G}$ is the basis with respect to purely lexicographic order induced by the variable ordering

$$x_{i_1} \succ \cdots \succ x_{i_k} \succ \text{(any order of variables other than} x_{i_1}, \dots, x_{i_k}.$$

If the output is "No", then any variables in X in this step appear in both the initial terms and the trailing terms of binomials in $\mathcal{G} \cap k[X]$. (*Trailing term* is the term of binomial which is not initial term.) Thus however we take the variables, the binomials $\mathcal{G} \cap k[X]$ remains. This shows that $\mathcal{G}$ is not the basis with respect to the purely lexicographic term order. ■

Example 4.7 *Let* $n = 5$*. Let*

$$\begin{aligned} \mathcal{G}_1 = \{ & x_{12}x_{23} - x_{13}, x_{12}x_{24} - x_{14}, x_{12}x_{25} - x_{15}, x_{13}x_{24} - x_{14}x_{23}, x_{13}x_{25} - x_{15}x_{23}, \\ & x_{13}x_{34} - x_{14}, x_{13}x_{35} - x_{15}, x_{14}x_{25} - x_{15}x_{24}, x_{14}x_{35} - x_{15}x_{34}, x_{14}x_{45} - x_{15}, \\ & x_{23}x_{34} - x_{24}, x_{23}x_{35} - x_{25}, x_{24}x_{35} - x_{25}x_{34}, x_{24}x_{45} - x_{25}, x_{34}x_{45} - x_{35}\}. \end{aligned}$$

If we first remove x_{12} *from* X*, then*

$$
\begin{aligned}
F \;=\; & \{x_{13}x_{24} - x_{14}x_{23}, x_{13}x_{25} - x_{15}x_{23}, x_{13}x_{34} - x_{14}, x_{13}x_{35} - x_{15}, x_{14}x_{25} - x_{15}x_{24}, \\
& x_{14}x_{35} - x_{15}x_{34}, x_{14}x_{45} - x_{15}, x_{23}x_{34} - x_{24}, x_{23}x_{35} - x_{25}, \\
& x_{24}x_{35} - x_{25}x_{34}, x_{24}x_{45} - x_{25}, x_{34}x_{45} - x_{35}\}.
\end{aligned}
$$

We next remove x_{13} *from* X*, and so on. As a result, when we remove the variables in the order* $x_{12}, x_{13}, x_{14}, x_{15}, x_{23}, x_{24}, x_{25}, x_{34}$*, then* F *become empty, and output "Yes". In fact,* $\mathcal{G}_1$ *is the reduced Gröbner basis with respect to the purely lexicographic order induced by the variable ordering*

$$x_{12} \succ x_{13} \succ x_{14} \succ x_{15} \succ x_{23} \succ x_{24} \succ x_{25} \succ x_{34} \succ x_{35} \succ x_{45}.$$

Let

$$
\begin{aligned}
\mathcal{G}_2 \;=\; & \{x_{12}x_{23} - x_{13}, x_{12}x_{24} - x_{14}, x_{12}x_{25} - x_{15}, x_{13}x_{24} - x_{14}x_{23}, x_{13}x_{34} - x_{14}, \\
& x_{13}x_{35} - x_{15}, x_{14}x_{45} - x_{15}, x_{15}x_{23} - x_{13}x_{25}, x_{15}x_{24} - x_{14}x_{25}, x_{15}x_{34} - x_{14}x_{35}, \\
& x_{23}x_{34} - x_{24}, x_{23}x_{35} - x_{25}, x_{24}x_{35} - x_{25}x_{34}, x_{24}x_{45} - x_{25}, x_{34}x_{45} - x_{35}\}.
\end{aligned}
$$

After we remove two variables x_{12} *and* x_{45} *from* X*,*

$$
\begin{aligned}
F \;=\; & \{x_{13}x_{24} - \underline{x_{14}x_{23}}, x_{13}x_{34} - x_{14}, x_{13}x_{35} - \underline{x_{15}}, x_{15}x_{23} - \underline{x_{13}x_{25}}, x_{15}x_{24} - x_{14}x_{25}, \\
& x_{15}x_{34} - x_{14}\underline{x_{35}}, x_{23}x_{34} - \underline{x_{24}}, x_{23}x_{35} - x_{25}, x_{24}x_{35} - x_{25}\underline{x_{34}}\}.
\end{aligned}
$$

Since variables in $X = \{x_{13}, x_{14}, x_{15}, x_{23}, x_{24}, x_{25}, x_{34}, x_{35}\}$ *appear in the trailing term of binomials in* F *(see lined variables), the output is "No".* ■

We implement Algorithm 4.6 in C and examine on $n = 5, 6$. Table 4.2 is the result for $n = 5$ and Table 4.3 is the result for $n = 6$. All the experiments were done on Sun UltraSPARC-II, 360 MHz workstation with 1GB memory.

These results show that a ratio of the bases with respect to purely lexicographic orders to all of the bases becomes small when the number of elements are large. And the result for $n = 6$ shows that maximum of the number of elements in reduced Gröbner bases with respect to purely lexicographic term orders is 36, which is smaller than that with respect to any term orders. It seems that for $n \geq 7$ maximum with respect to purely lexicographic orders is smaller than that with respect to any term orders.

# elements	6	7	8	9	10	11	12	13	14	15
# GB	22	20	45	26	23	28	19	10	0	18
# GB w.r.t. lex	22	20	45	26	23	27	17	8	0	16

Table 4.2: The number of reduced Gröbner bases (# GB) and the number of bases with respect to lexicographic orders (# GB w.r.t. lex) for D_5.

# elements	10	11	12	13	14	15	16
# GB	90	140	487	585	857	1483	1776
# GB w.r.t. lex	90	140	487	585	857	1466	1700
# elements	17	18	19	20	21	22	23
# GB	2062	2158	3212	3279	3173	3015	3215
# GB w.r.t. lex	1910	1914	2736	2567	2263	2061	2331
# elements	24	25	26	27	28	29	30
# GB	3408	3710	2860	2091	1383	2621	2393
# GB w.r.t. lex	2012	2165	1266	915	736	1422	1018
# elements	31	32	33	34	35	36	37
# GB	1440	754	204	0	1364	508	44
# GB w.r.t. lex	492	154	0	0	736	64	0

Table 4.3: The number of reduced Gröbner bases (# GB) and the number of bases with respect to lexicographic orders (# GB w.r.t. lex) for D_6.

Question 4.8 *What would be the maximum number of elements with respect to the purely lexicographic orders?*

We show an example of reduced Gröbner basis for I_{A_6} with respect to purely lexicographic order whose number of elements is 36, more than that of the basis in Theorem 3.11.

Example 4.9 ([15]) *Let $n = 6$. The reduced Gröbner basis for I_{A_6} with respect to the purely lexicographic order induced by the variable ordering*

$$x_{12} \succ x_{13} \succ x_{23} \succ x_{45} \succ x_{46} \succ x_{56} \succ x_{14} \succ x_{25}$$

$$\succ x_{36} \succ x_{15} \succ x_{16} \succ x_{24} \succ x_{26} \succ x_{34} \succ x_{35}$$

has 36 binomials as the following.

$$\{\underline{x_{12}x_{23}} - x_{13}, \underline{x_{12}x_{24}} - x_{14}, \underline{x_{12}x_{25}} - x_{15}, \underline{x_{12}x_{26}} - x_{16}, \underline{x_{13}x_{24}} - x_{14}x_{23},$$

$$\underline{x_{13}x_{25}} - x_{15}x_{23}, \underline{x_{13}x_{26}} - x_{16}x_{23}, \underline{x_{13}x_{34}} - x_{14}, \underline{x_{13}x_{35}} - x_{15}, \underline{x_{13}x_{36}} - x_{16},$$

$$\underline{x_{14}x_{25}} - x_{15}x_{24}, \underline{x_{14}x_{26}} - x_{16}x_{24}, \underline{x_{14}x_{35}} - x_{15}x_{34}, \underline{x_{14}x_{36}} - x_{16}x_{34}, \underline{x_{14}x_{45}} - x_{15},$$

$$\underline{x_{14}x_{46}} - x_{16}, \underline{x_{15}x_{36}} - x_{16}x_{35}, \underline{x_{15}x_{56}} - x_{16}, \underline{x_{16}x_{25}} - x_{15}x_{26}, \underline{x_{16}x_{45}} - x_{15}x_{46},$$

$$\underline{x_{23}x_{34}} - x_{24}, \underline{x_{23}x_{35}} - x_{25}, \underline{x_{23}x_{36}} - x_{26}, \underline{x_{24}x_{36}} - x_{26}x_{34}, \underline{x_{24}x_{45}} - x_{25},$$

$$\underline{x_{24}x_{46}} - x_{26}, \underline{x_{25}x_{34}} - x_{24}x_{35}, \underline{x_{25}x_{36}} - x_{26}x_{35}, \underline{x_{25}x_{56}} - x_{26}, \underline{x_{26}x_{45}} - x_{25}x_{46},$$

$$\underline{x_{34}x_{45}} - x_{35}, \underline{x_{34}x_{46}} - x_{36}, \underline{x_{35}x_{56}} - x_{36}, \underline{x_{36}x_{45}} - x_{35}x_{46}, \underline{x_{45}x_{56}} - x_{46}, \underline{x_{15}x_{26}x_{34}} - x_{16}x_{24}x_{35}\}$$

Chapter 5

Applications to Integer Programming

In this chapter we apply the toric ideals I_{A_n} to the minimum cost flow problem. Conti and Traverso [5] introduced an algorithm based on Gröbner basis to solve integer programs. There are several results which apply the toric ideals of graphs to integer programs [8, 9]. We first describe two versions of Conti-Traverso algorithm, one is the original version [5] and the other is a condensed version [23]. We next apply the toric ideals I_{A_n} to the minimum cost flow problem using Conti-Traverso algorithm. However, the complexity of Conti-Traverso algorithm is not known.

5.1 Conti-Traverso Algorithm

In this section, we describe Conti-Traverso algorithm [5]. Let $A \in \mathbb{Z}^{d\times n}$, $b \in \mathbb{Z}^d$, $c \in \mathbb{R}^n_{\geq 0}$. We consider the integer program

$$IP_{A,c}(b) := minimize\{c \cdot x \colon Ax = b,\ x \in \mathbb{N}^n\}.$$

Conti-Traverso algorithm is the algorithm which solves $IP_{A,c}(b)$ using the toric ideal I_A.

We first describe the condensed version of Conti-Traverso algorithm [20, 23]. This version is useful for highlighting the main computational step involved, but there are number of issues that have to be dealt with while implementing.

Algorithm 5.1 (The Conti-Traverso Algorithm [20, 23])
Input: *$A \in \mathbb{Z}^{d\times n}$, $b \in \mathbb{Z}^d$, $c \in \mathbb{R}^n_{\geq 0}$*
Output: *An optimal solution $\mathbf{u}'$ for $IP_{A,c}(b)$*
1. *Compute the reduced Gröbner basis $\mathcal{G}_{\succ_c}$ of I_A.*

2. *For any solution* $\mathbf{u}$ *of* $IP_{A,\succ_c}(b)$*, compute the normal form* $\mathbf{x}^{\mathbf{u}'}$ *of* $\mathbf{x}^{\mathbf{u}}$ *by* $\mathcal{G}_{\succ_c}$*.*

3. *Output* $\mathbf{u}'$*.* $\mathbf{u}'$ *is the unique optimal solution of* $IP_{A,c}(b)$*.*

Example 5.2 ([23, Example 2.5]) *Consider the matrix*

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \end{pmatrix}$$

and the cost vector $c = (1,5,5,1,0)$*. Let* $\succ_c$ *be a refinement of* c *with respect to purely lexicographic order induced by* $x_1 \succ x_2 \succ x_3 \succ x_4 \succ x_5$*. The reduced Gröbner basis for* I_A *with respect to* $\succ_c$ *is*

$$\mathcal{G}_c = \{\underline{x_3x_5^3} - x_1^2x_4^2, \underline{x_2x_5^2} - x_1^2x_4, \underline{x_2x_4} - x_3x_5, \underline{x_2^2x_5} - x_1^2x_3\}.$$

For $b = (25,34,18)$*,* $(1,10,10,4,0)$ *is a solution of* $IP_{A,c}(b)$*. The normal form of the monomial* $x_1x_2^{10}x_3^{10}x_4^4$ *with respect to* $\mathcal{G}_c$ *is* $x_1^7x_3^{17}x_5$ *and hence the optimal solution of* $IP_{A,c}(b)$ *is* $(7,0,17,0,1)$*.* ■

There are some issues that have to be dealt with while implementing Algorithm 5.1. In particular, finding a generating set for I_A to be used as input to the Buchberger algorithm in Step 1 and finding initial solution $\mathbf{u}$ of $IP_{A,c}(b)$ are non-trivial. This version is highlighting the main computational steps, although the original version is more friendly to implement. In particular, the original algorithm [5] uses a single Gröbner basis computation to achieve Step 1. of Algorithm 5.1, and finds initial solution $\mathbf{u}$ automatically.

Algorithm 5.3 (The Conti-Traverso Algorithm [5])

Input: $A \in \mathbb{Z}^{d\times n}$*,* $b \in \mathbb{Z}^d$*,* $c \in \mathbb{R}^n_{\geq 0}$

Output: *An optimal solution* $\mathbf{u}'$ *for* $IP_{A,c}(b)$

0. *Consider the ideal* $J = \langle x_1\mathbf{t}^{\mathbf{a}_1^-} - \mathbf{t}^{\mathbf{a}_1^+}, \dots, x_n\mathbf{t}^{\mathbf{a}_n^-} - x_n\mathbf{t}^{\mathbf{a}_n^+}, t_0t_1\cdots t_d - 1\rangle$ *in the polynomial ring* $k[x_1,\dots,x_n,t_0,t_1,\dots,t_d]$*. Let* $t_{\bar{0}} = \{t_1,\dots,t_d\}$*.*

1. *Compute the reduced Gröbner basis* $\mathcal{G}_{\succ'}$ *of* J *with respect to any elimination order* $\succ'$ *such that* $\{t_0,t_1,\dots,t_d\} \succ' \{x_1,\dots,x_n\}$ *and* $\succ'$ *restricted to* $k[x_1,\dots,x_n]$ *induces the same total order as* $\succ_c$*.*

2. *In order to solve* $IP_{A,\succ_c}(b)$*, form the monomial* $\mathbf{t}^b = t_0^{\beta}t_{\bar{0}}^{b+\beta(\mathbf{e}_1+\cdots+\mathbf{e}_d)}$ *where* $\beta = \max\{|b_j| : b_j < 0\}$ *and* $\mathbf{e}_i$ *is the* i*-th unit vector in* $\mathbb{R}^d$*. Compute the normal* *form* $\mathbf{t}^{\gamma}\mathbf{x}^{\mathbf{u}'}$ *of the monomial* $\mathbf{t}^{b}$ *with respect to* $\mathcal{G}_{\succ'}$*.*

3. *If* $\gamma = 0$ *then* $IP_{A,\succ_c}(b)$ *is feasible with optimal solution* $\mathbf{u}'$*. Else* $IP_{A,\succ_c}(b)$ *is infeasible.*

As we described in Remark 2.44, if all entries of the matrix A are non-negative, we do not need the variable t_0 and the binomial $t_0 t_1 \cdots t_d - 1$ in the above algorithm.

5.2 Applications to Minimum Cost Flow Problem

In this section, we consider an application of the Gröbner bases of I_{A_n} to (uncapacitated) minimum cost flow problem on D_n. The *minimum mean cycle-canceling algorithm* [10] is known as strongly polynomial time algorithm which depends only on the number of vertices and edges. In this algorithm, if the mean cost of a directed cycle in the residual network is negative, the algorithm cancels flows along this cycle. But since the cycles which the algorithm may choose to cancel are all of the cycles in the network, its number is of exponential. We show that, for the network D_n, we may choose the cycle from the cycle corresponding reduced Gröbner basis, and the minimum cost flow can be computed by canceling flows along the cycle similarly. But the time complexity of this method is unknown.

For the minimum cost flow problem, we refer to [1].

5.2.1 Introduction

Let $G = (V, A)$ be a directed graph with a cost $c_{ij} \in \mathbb{R}_{>0}$ associated with every edge $(i, j) \in A$. We associate with each vertex $i \in V$ a number $b(i) \in \mathbb{N}$ which indicates its supply when $b(i) > 0$, or its demand when $b(i) < 0$. We assume that $\sum_{i \in V} b(i) = 0$. The (uncapacitated) minimum cost flow problem can be stated as follows:

$$\textit{Minimize} \quad z(x) = \sum_{(i,j)\in A} c_{ij} x_{ij} \tag{5.1}$$

$$\textit{s.t.} \quad \sum_{j:\,(i,j)\in A} x_{ij} - \sum_{j:\,(j,i)\in A} x_{ji} = b(i), \quad \text{for all } i \in V. \tag{5.2}$$

Let x^*_{ij} be a feasible solution which satisfies (5.2). Then we define the *residual network* (V, A') as follows. A' consists of the directed edges (i, j) and (j, i). Each edge (i, j) has cost c_{ij}, and each edge (j, i) has cost $-c_{ij}$ and *residual capacity* $r_{ji} = x^*_{ij}$ i.e. we can flow from j to i less than r_{ji}.

5.2.2 Minimum Mean Cycle-canceling Algorithm

We introduce the *minimum mean cycle-canceling algorithm* [10] for minimum cost flow problem. This algorithm is known to be strongly polynomial time algorithm.

We define the *mean cost* of a directed cycle W to be $\left(\sum_{(i,j)\in W} c_{ij}\right) / |W|$, and the *minimum mean cycle* to be a cycle with the smallest mean cost in the network.

Algorithm 5.4 (The minimum mean cycle-canceling algorithm [10])
compute a feasible flow x which satisfies (5.2)
while *there exists a minimum mean cycle W in the residual network of x whose mean cost is negative* **do**
flow along W and update a feasible flow x
output an optimal flow x

Theorem 5.5 (See [1, Theorem 10.16.]) *Let n be the number of vertices and m the number of edges. The minimum mean cycle-canceling algorithm performs $O(nm^2 \log n)$ iterations and runs in $O(n^2 m^3 \log n)$ time.*

But since the cycles which the algorithm can choose to augment are all of cycles in the network, its number is of exponential.

5.2.3 Conti-Traverso Algorithm for Minimum Cost Flow Problem

Let the network (V, A) be the acyclic tournament graph D_n. Then we can write (5.2) as $A_n x = {}^t(b(1), b(2), \dots, b(n))$, where A_n is the vertex-edge incidence matrix of D_n. Thus we can apply the Conti-Traverso algorithm. If reduced Gröbner basis with respect to the term order which corresponds to the cost vector is known, we can obtain the minimum cost flow by canceling the cycles which correspond to the elements in reduced Gröbner basis. If we ignore the time calculating the reduced Gröbner basis, the running time is corresponding only to the reduction by the reduced Gröbner basis. But the upper bound for the number of reduction is not known.

Example 5.6 *Let* $n = 4$*. Let* $c := (c_{12}, c_{13}, c_{14}, c_{23}, c_{24}, c_{34}) = (100000, 10000, 1000, 100, 10, 1)$ *and* $b = {}^t(5, 2, -3, -4)$*. In this case, the reduced Gröbner basis for* I_{A_4} *is*

$$\mathcal{G} = \{\underline{x_{12}x_{23}} - x_{13}, \underline{x_{12}x_{24}} - x_{14}, \underline{x_{13}x_{34}} - x_{14}, \underline{x_{23}x_{34}} - x_{24}, \underline{x_{13}x_{24}} - x_{14}x_{23}\}.$$

Let the initial feasible flow be $\mathbf{u} = (3, 0, 2, 3, 2, 0)$ *(Figure 5.2 left). First we cancel*

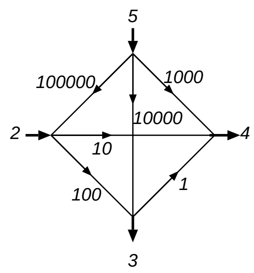

Figure 5.1: Minimum cost flow problem in Example 5.6

the flow along the cycle $1, 3, 2, 1$*, then we get the improved flow* $\mathbf{u}_1 = (0, 3, 2, 0, 2, 0)$ *(Figure 5.2 center). Last we cancel the flow along the cycle* $1, 4, 2, 3, 1$*, then we get the minimum cost flow* $\mathbf{u}_2 = (0, 1, 4, 2, 0, 0)$ *(Figure 5.2 right). In Conti-Traverso*

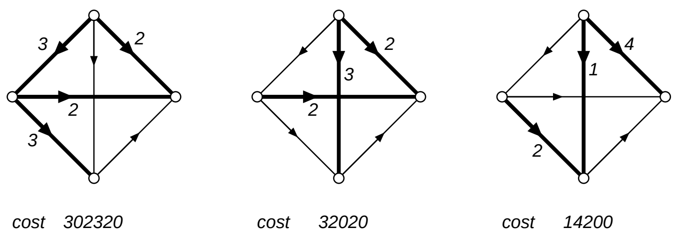

Figure 5.2: The initial flow $\mathbf{u}$ (left), the improved flow $\mathbf{u}_1$ (center), the minimum cost flow $\mathbf{u}_2$ (right).

algorithm, these steps correspond reducing $x_{12}^3 x_{14}^2 x_{23}^3 x_{24}^2$ *by* $\mathcal{G}$:

$$\begin{aligned} x_{12}^3 x_{14}^2 x_{23}^3 x_{24}^2 &\xrightarrow{x_{12}x_{23}-x_{13}} x_{13}^3 x_{14}^2 x_{24}^2 \\ &\xrightarrow{x_{13}x_{24}-x_{14}x_{23}} x_{13} x_{14}^4 x_{23}^2 \end{aligned}$$

where

$$x_{12}^3 x_{14}^2 x_{23}^3 x_{24}^2 \xrightarrow{x_{12}x_{23}-x_{13}} x_{13}^3 x_{14}^2 x_{24}^2$$

means that the normal form of $x_{12}^3 x_{14}^2 x_{23}^3 x_{24}^2$ *by* $\underline{x_{12}x_{23}} - x_{13}$ *equals* $x_{13}^3 x_{14}^2 x_{24}^2$. ■

Chapter 6

Conclusion and Future Work

Gröbner bases are applied to some computationally hard problems in recent years. On the other hand, the properties of graphs may give insight for Gröbner bases of toric ideals. Toric ideals of undirected complete graphs and bipartite graphs have been studied, but those of other graphs are not well understood. We have studied Gröbner bases for toric ideals of acyclic tournament graphs.

We have given the positive grading for which the toric ideal becomes homogeneous, and shown the reduced Gröbner bases of toric ideals with respect to some term orders when the positive grading is standard grading or graphical grading. All of the bases we have shown has polynomial size. And we showed the experimental result in the case of graphical grading. But the upper bound for the number of elements are not known. For bounding the degree of reduced Gröbner bases, we showed the minimum degree and the maximum degree in the case of standard grading, and showed the lower bound and upper bound in the case of graphical grading.

We also showed the application to minimum cost flow problems. The minimum mean cycle-canceling algorithm is known for minimum cost flow problem. We showed the relation of this algorithm with the Conti-Traverso algorithm for acyclic tournament graphs defining the network. Main step of this algorithm was canceling feasible flow by negative cycle, which corresponds to reducing monomial by reduced Gröbner basis. But the complexity of canceling cycles are not known. To study the effectiveness of this application, we need to analyze the number of elements of reduced Gröbner bases and the complexity of canceling cycles, which should be future works.

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