Computational Complexity of Ordinary Differential Equations

Akitoshi KAWAMURA

University of Tokyo

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Problem

Given $g\colon [0,1] \times \mathbf{R} \to \mathbf{R}$, consider the ODE

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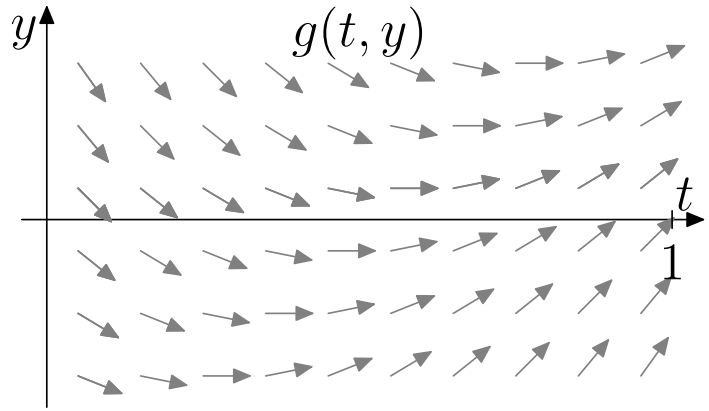

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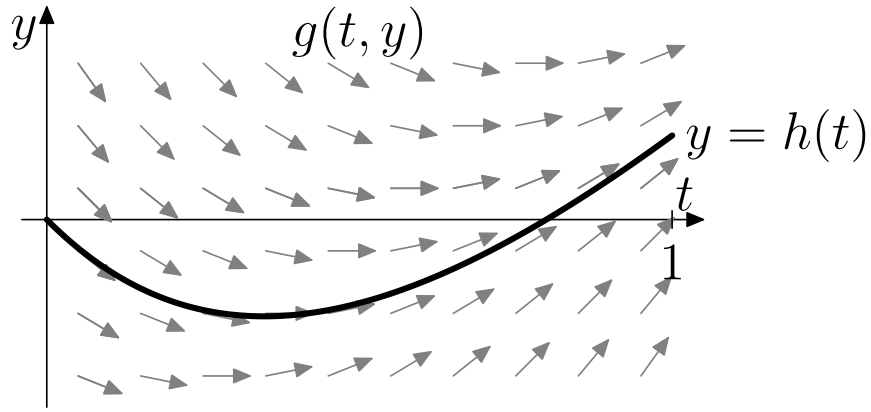

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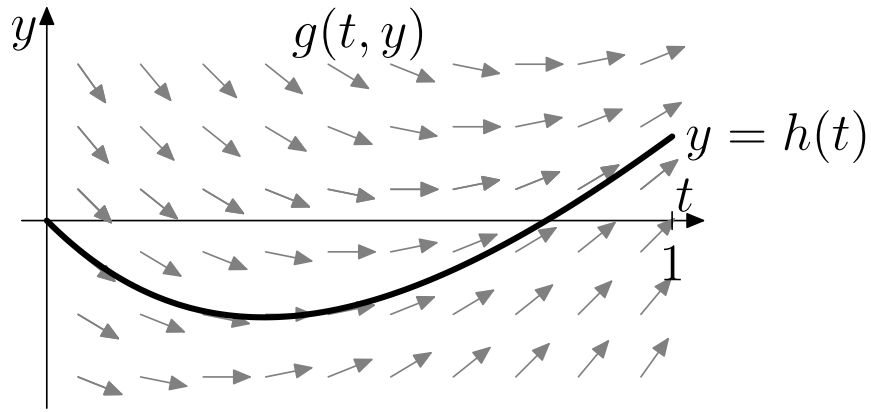

Question

g is simple $\Longrightarrow$ h is simple? (in terms of Computational Complexity)

Summary: Complexity of ODE solutions

$$h(0) = 0, \qquad h'(t) = g\big(t, h(t)\big)$$

Assuming g is polytime, how complex can h be?

Assumptions	Upper bound	Lower bound
None	——	can be (non-unique and) all non-computable [Pour-El 1979]
h is the unique solution	computable [implicit in Osgood 1898]	can take arbitrarily long time (or space) [Miller 1970]
g is 'locally' Lipschitz [Ko 1992]	exponential space [Ko 1992]	can be **EXPSPACE**-hard [K 2010]
g is Lipschitz	polyspace [Ko 1983]	can be **PSPACE**-hard [K 2010] (CCC 2009)
g is of class C^1		can be **PSPACE**-hard [K, Ota, Rösnick, Ziegler 2012] (MFCS 2012)
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Outline

$$h(0) = 0, \qquad h'(t) = g\big(t, h(t)\big)$$

1. Some complexity classes
   P, **NP**, **#P**, **PSPACE**, ...
2. Complexity of real functions
   How we define polytime real functions
3. Warm-up: Integration
   Assuming g is polytime and ignores the second argument, how complex can h be?
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   Assuming g is polytime and Lipschitz continuous, how complex can h be?
5. Final remarks
   Uniform versions / Summary

Outline

$$h(0) = 0, \qquad h'(t) = g\big(t, h(t)\big)$$

P the class of functions $A\colon \Sigma^* \to \{0,1\}$ such that there is a polynomial-time TM M satisfying

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PSPACE analogously, with polynomial-space.

NP and #P

NP

the class of functions $A\colon \Sigma^* \to \{0,1\}$ such that there are a polynomial p and a polytime TM M satisfying

$$A(x) = \begin{cases} 1 & \text{if there is } y \in \Sigma^{p(|x|)} \text{ with } M(x,y) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

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Tell whether a given (propositional) formula $\varphi(\vec{x})$ is satisfiable.

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Tell how many assignments satisfy the given formula $\varphi(\vec{x})$.

NP and #**P** relative to oracle B

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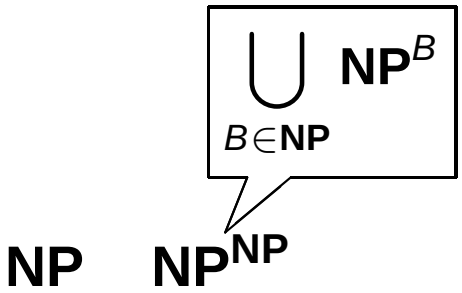

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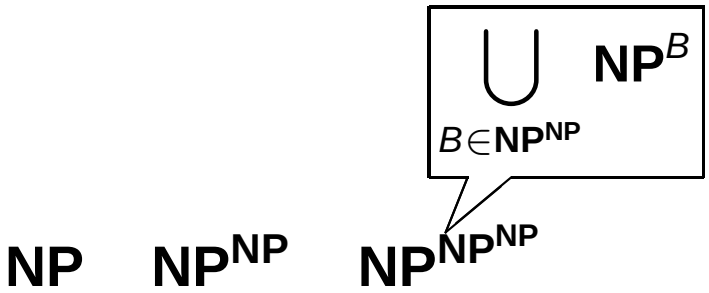

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General belief: These hierarchies do not collapse.
But we don't even have a proof of $\mathbf{P} \neq \mathbf{PSPACE}$.

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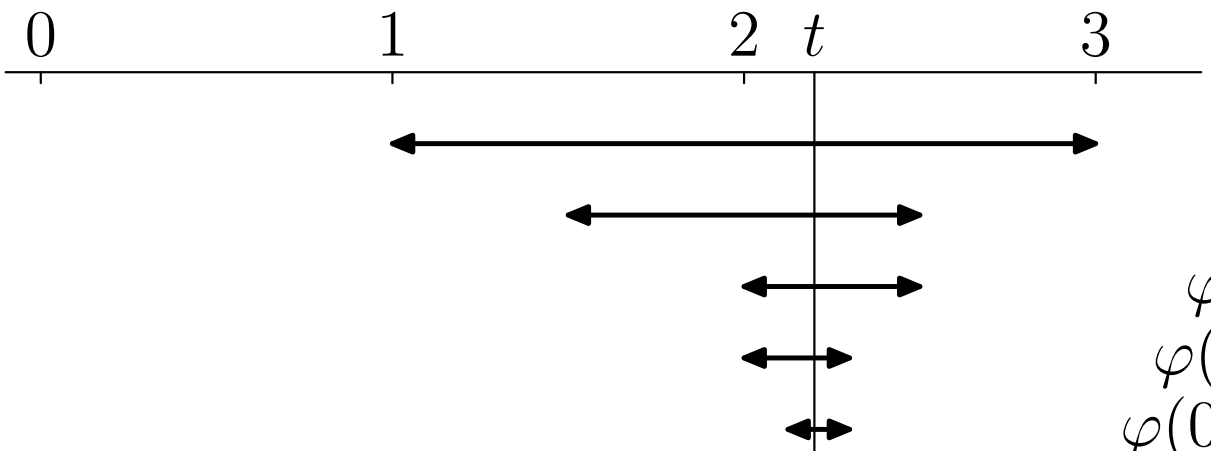
Representation of real numbers

Definition

We say that $\varphi\colon \Sigma^* \to \Sigma^*$ is a name of $t \in \mathbf{R}$ if for each $n \in \mathbf{N}$, $\varphi(0^n)$ is (the binary expansion of) t rounded up or down at the nth bit below the point.



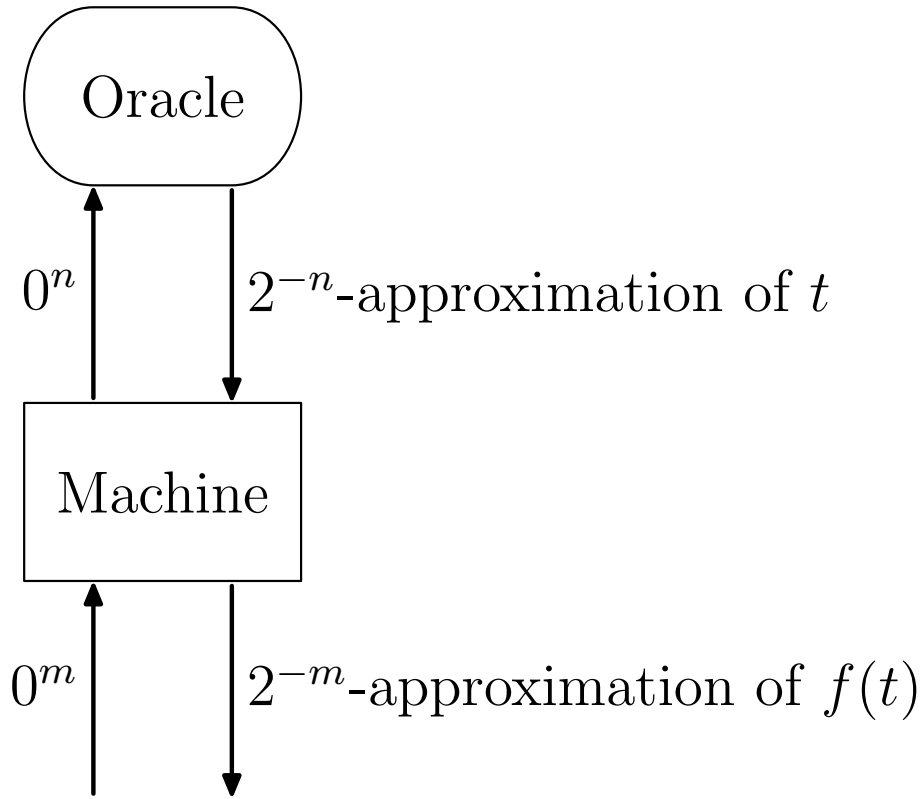
$$\begin{aligned}
\varphi(\varepsilon) &= +10. \\
\varphi(0) &= +10.0 \\
\varphi(00) &= +10.01 \\
\varphi(000) &= +10.001 \\
\varphi(0000) &= +10.0011 \\
&\vdots
\end{aligned}$$

(Each $\varphi(0^n)$ approximates t with precision 2^{-n})

Computing real functions

Definition

Machine M computes $f : [0,1] \to \mathbf{R}$ if, for any name φ of any $t \in [0,1]$, M^{φ} computes a name of $f(t)$.



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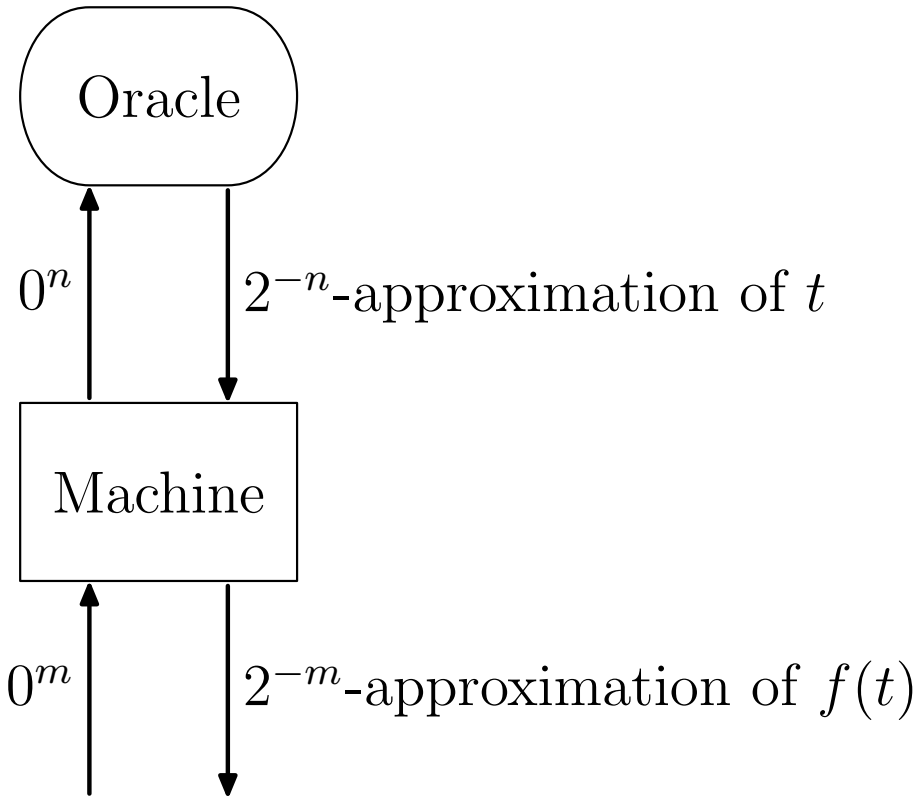

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- M Turing-reduces $f(t)$ to t.
- Given access to approximations of t to any precision, the machine yields $f(t)$ to any precision.

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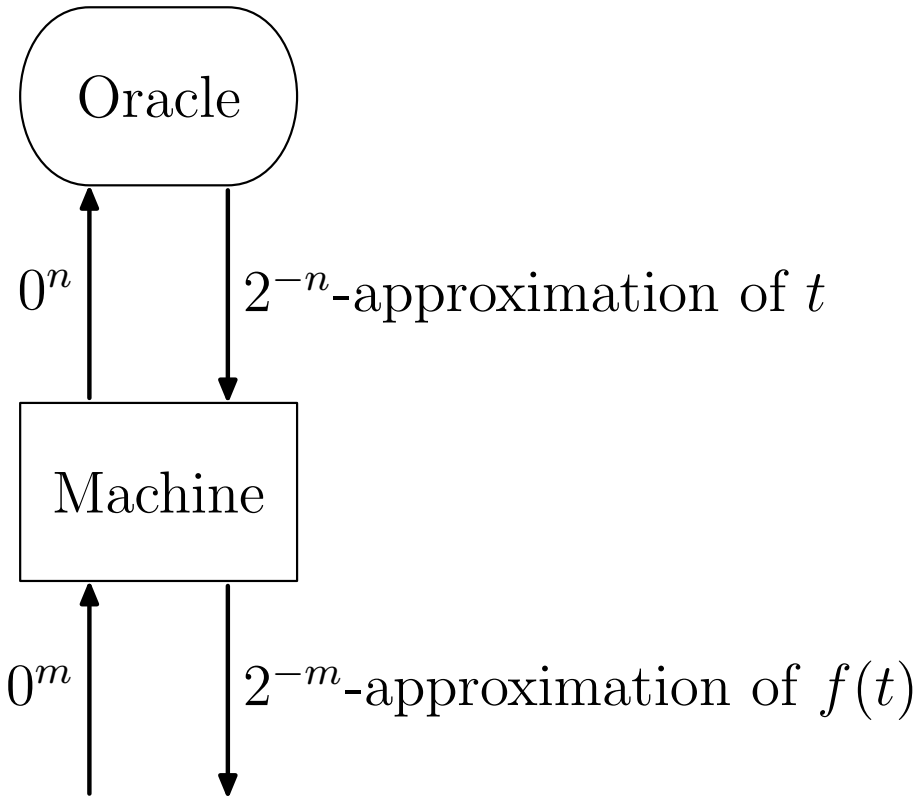

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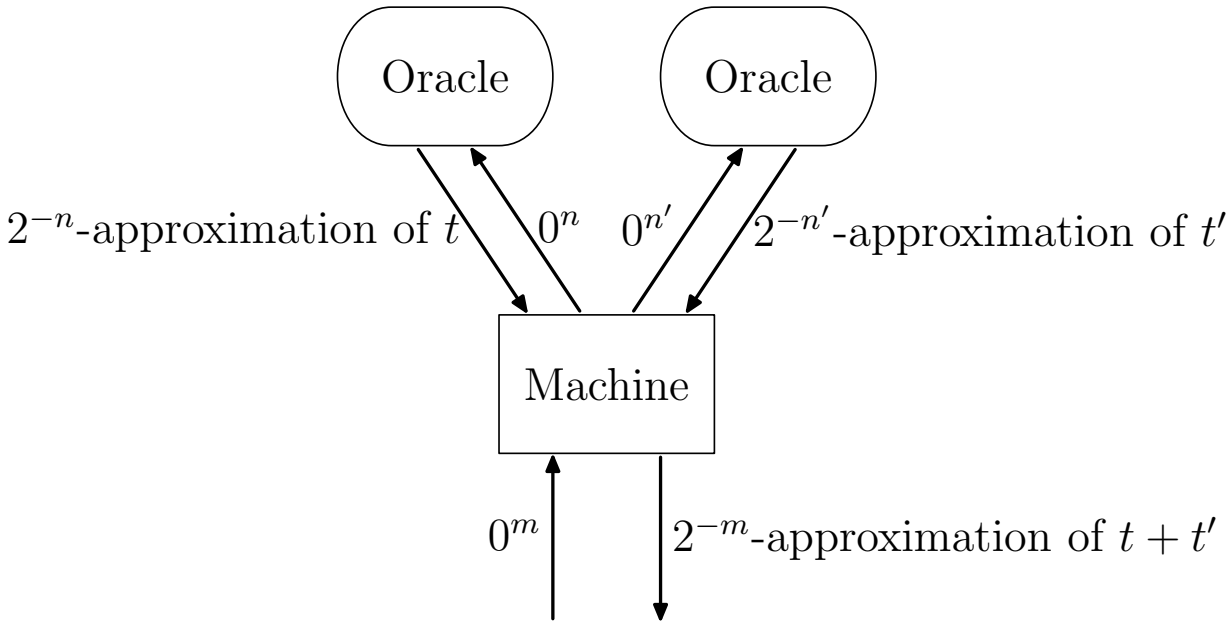
Note: Saying M is polytime means that it halts in time poly(m).
In particular, $n \leq$ poly(m).

Example 1

Addition $+\colon [0,1] \times [0,1] \to \mathbf{R}$ is polytime.

To do: Given (names of) t and t' as oracles and 0^m as input, output a 2^{-m}-approximation of $t + t'$.



- Ask the oracles for 2^{-m-2}-approximations of t and t'.
- Add the answers (as rational numbers).
- Output the closest rational number to the sum that has m bits below the point.

Example 2

$\exp\colon [0,1] \to \mathbf{R}$ is polytime.

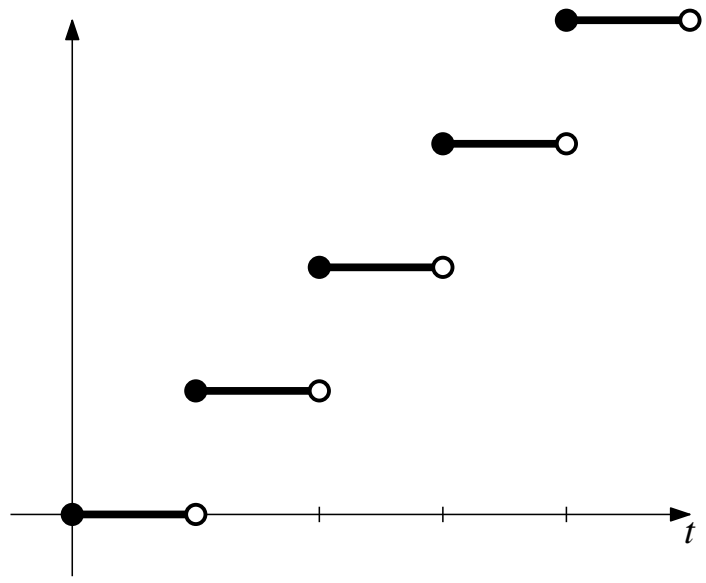
To get a 2^{-m}-approximation of the value

$$\exp t = \frac{1}{0!} + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots,$$

it suffices to compute a $2^{-m}/2$-approximation of the sum of the first m terms (because the remaining terms add up to at most $2^{-m}/2$).

Example 3

$\lfloor \cdot \rfloor : [0, 5] \to \mathbf{R}$ is not computable.



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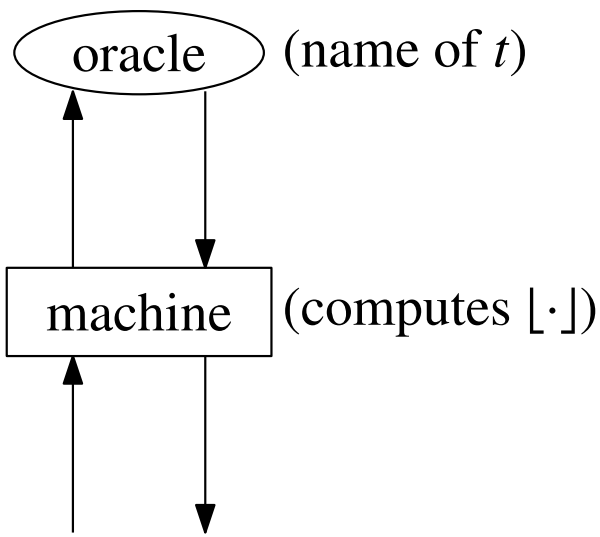

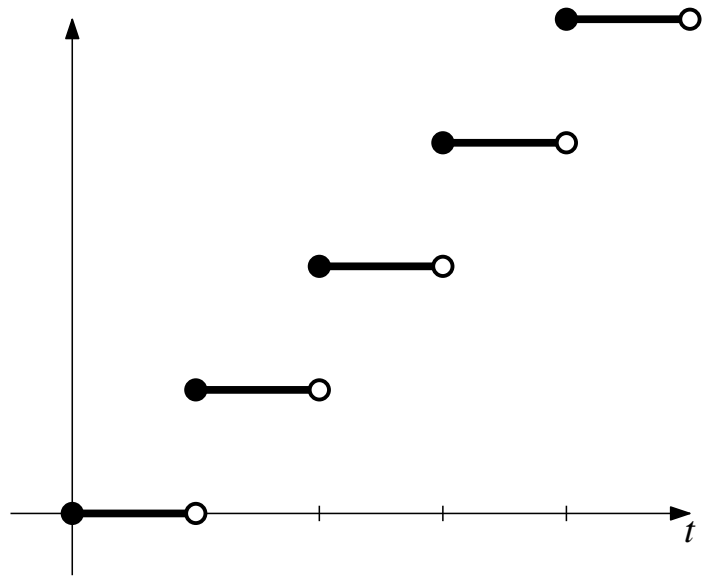

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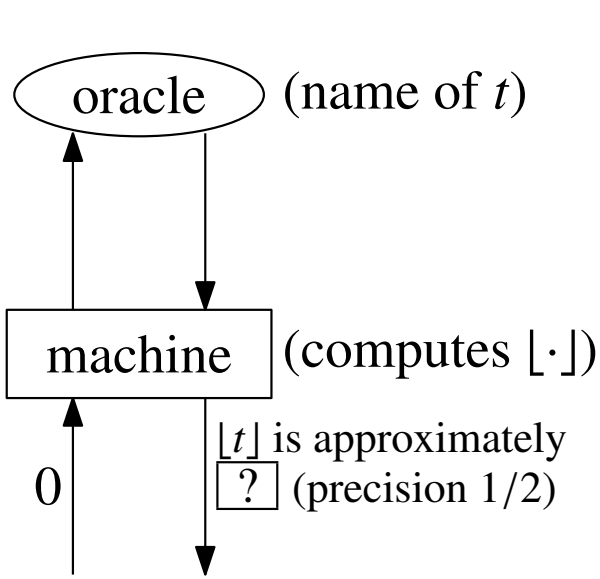

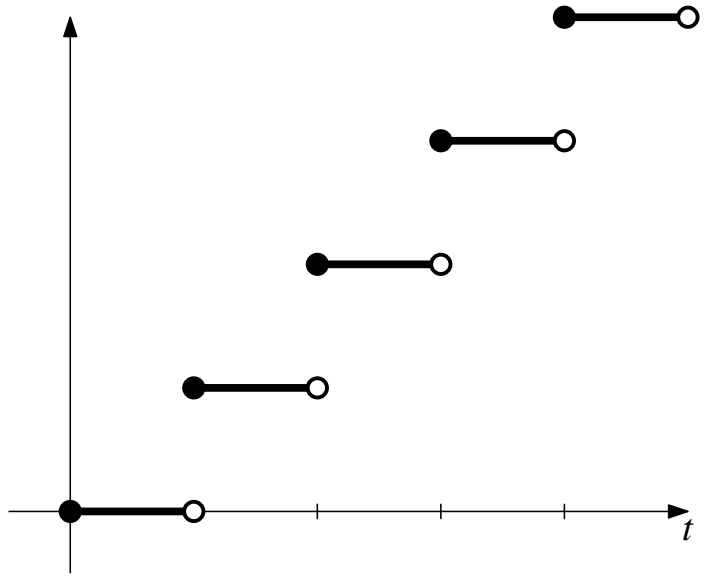

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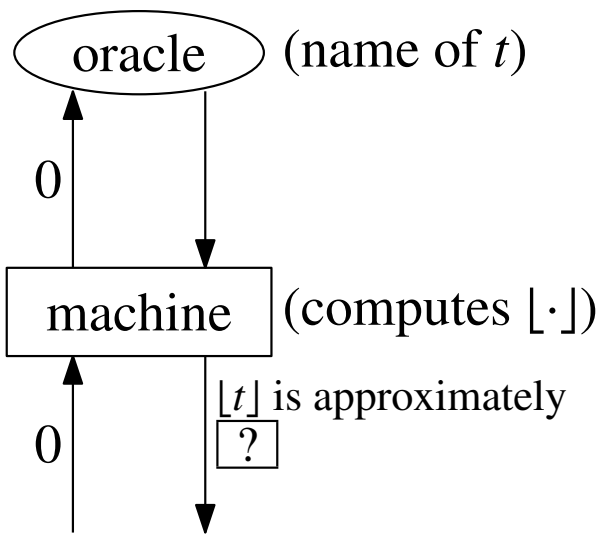

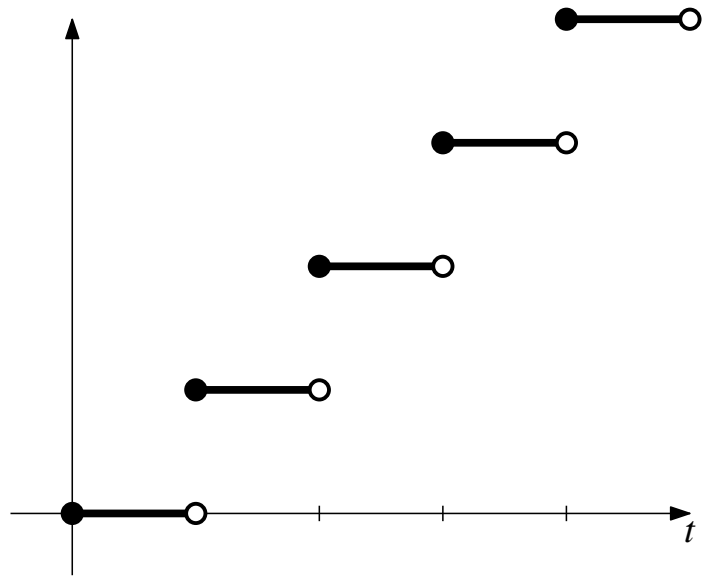

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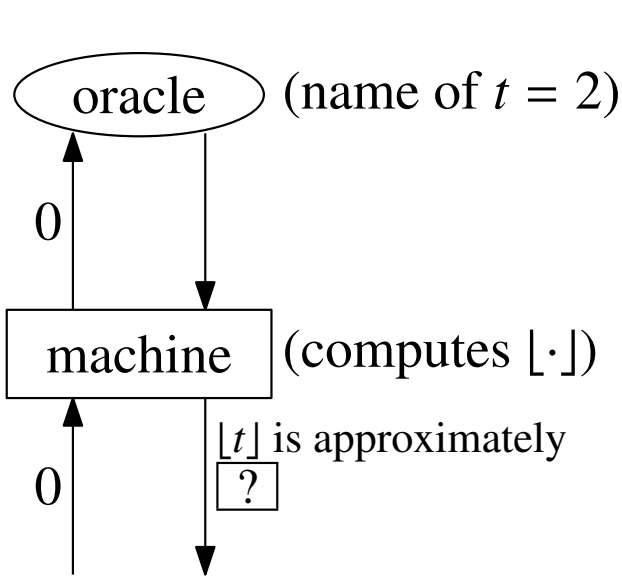

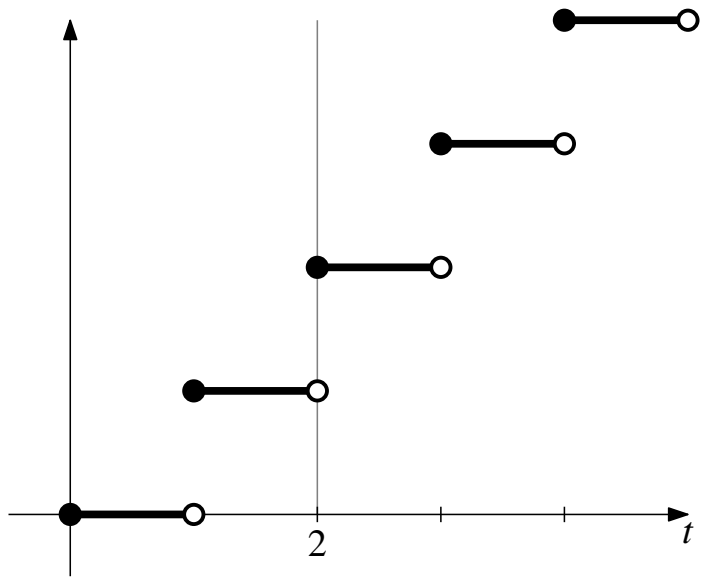

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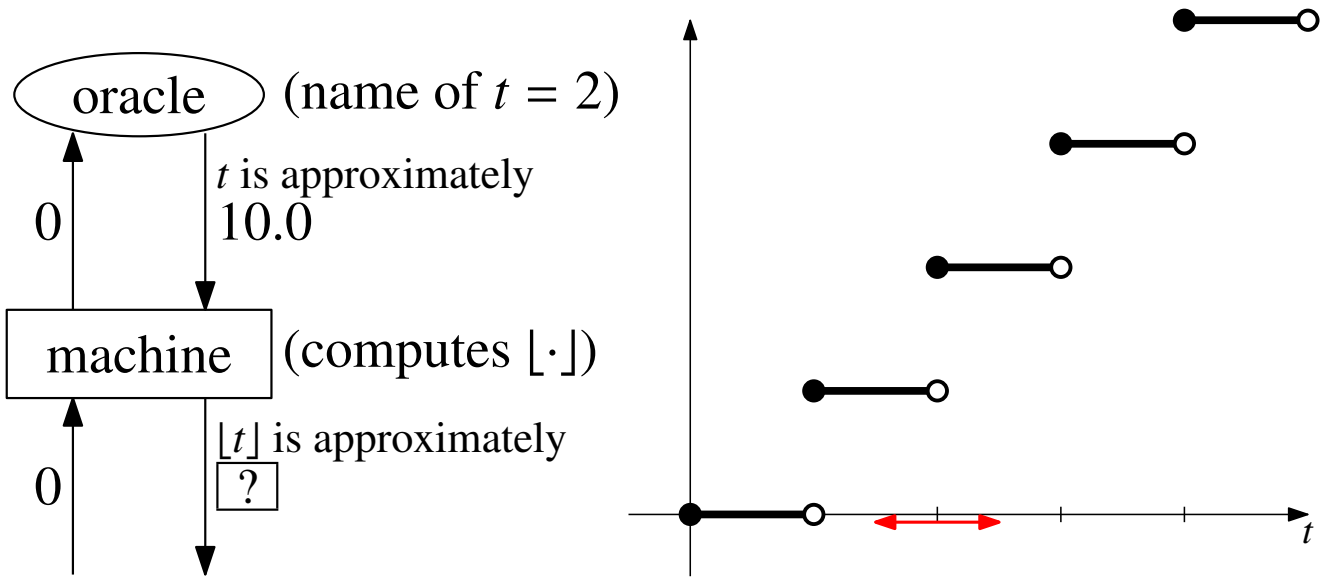

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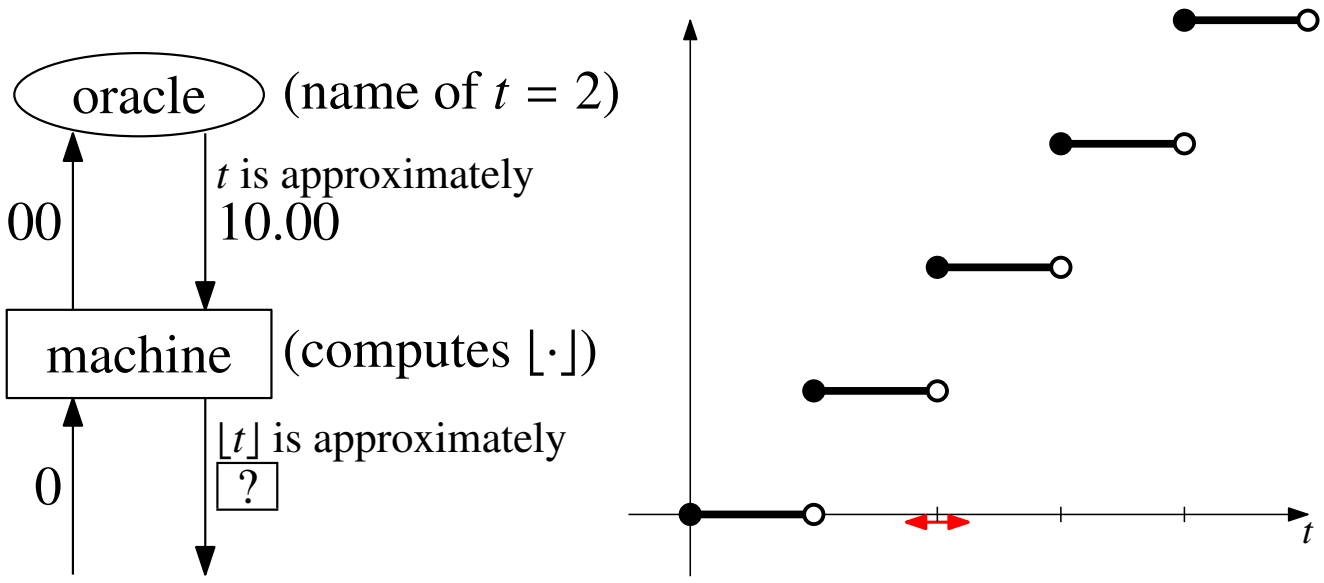

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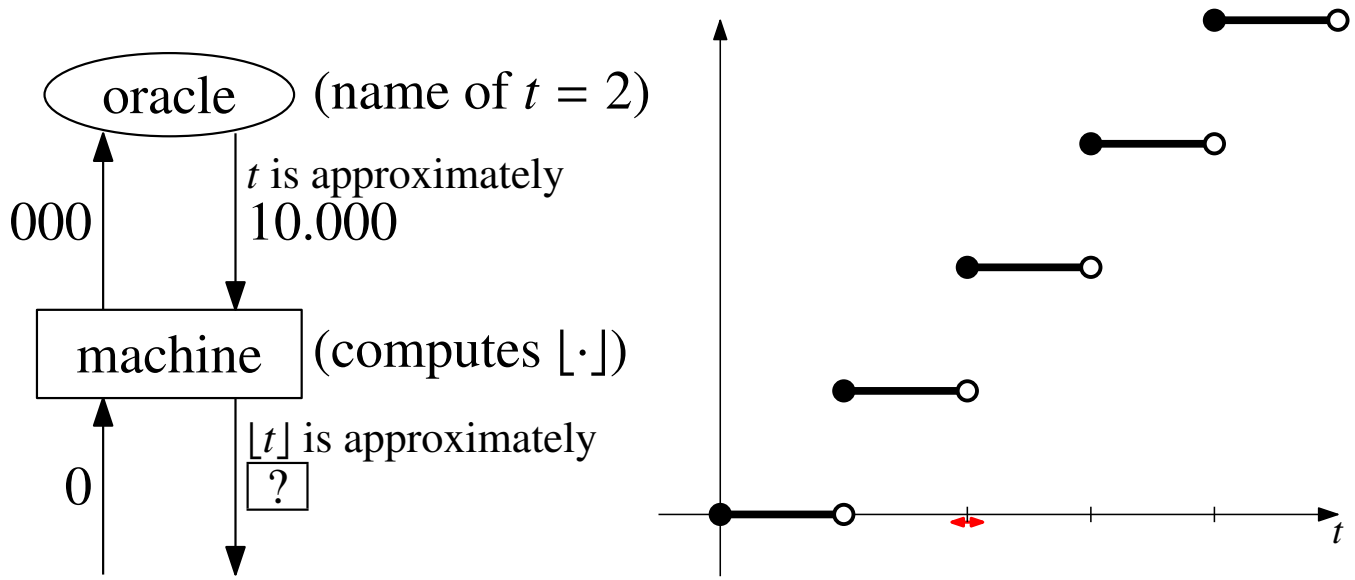

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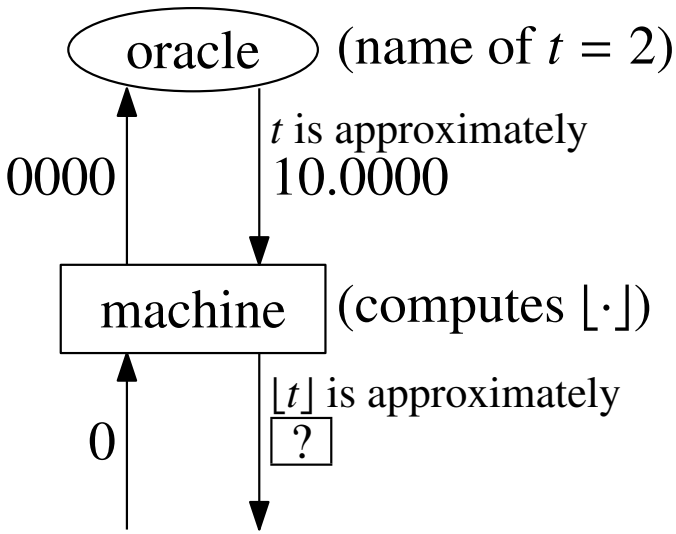
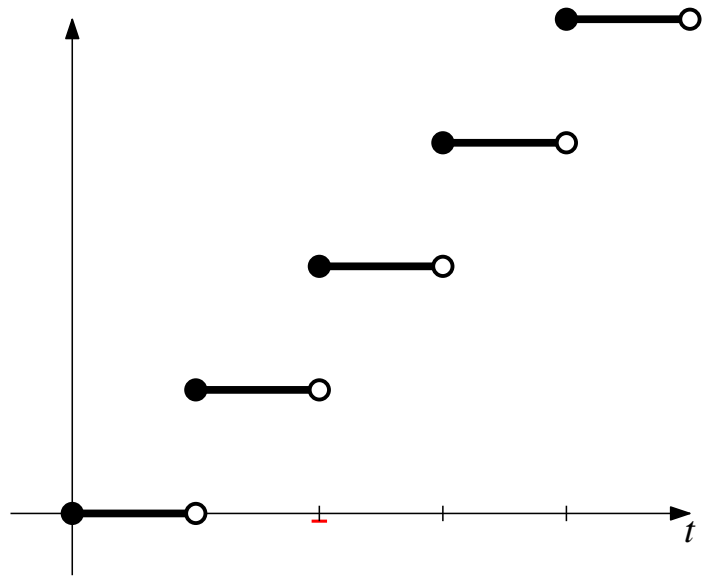




Computable functions are continuous

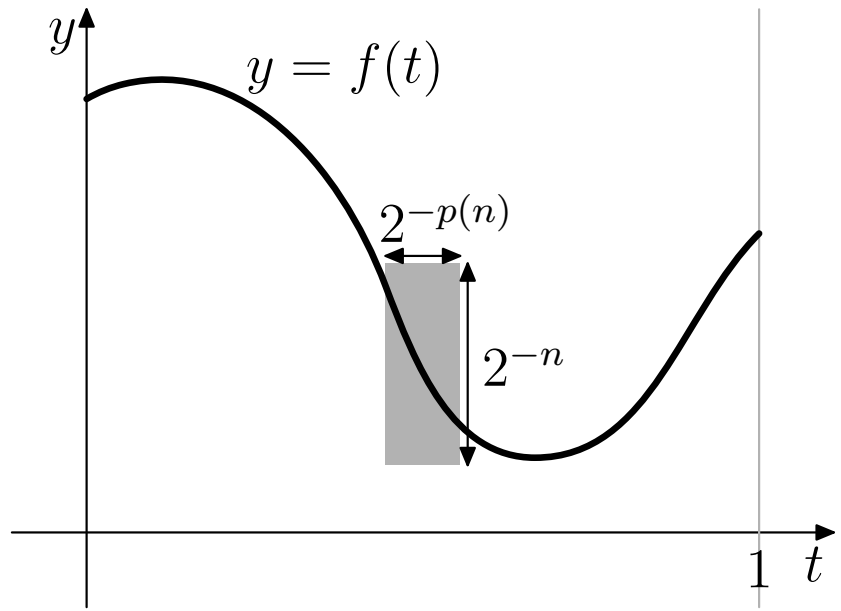
Theorem

- A computable real function is continuous.
- A polytime real function has a polynomial modulus of continuity.

Modulus of continuity p:

$|t - t'| < 2^{-p(n)}$

$\Longrightarrow |f(t) - f(t')| < 2^{-n}$.



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$f : [0,1] \to \mathbf{R}$ is polytime iff

- f has a polynomial modulus of continuity, and
- there is a function $\varphi : (\mathbf{Q} \cap [0,1]) \times \{0\}^* \to \mathbf{Q}$ in **FP** such that $|\varphi(u, 0^n) - f(u)| \leq 2^{-n}$ for each u and n.

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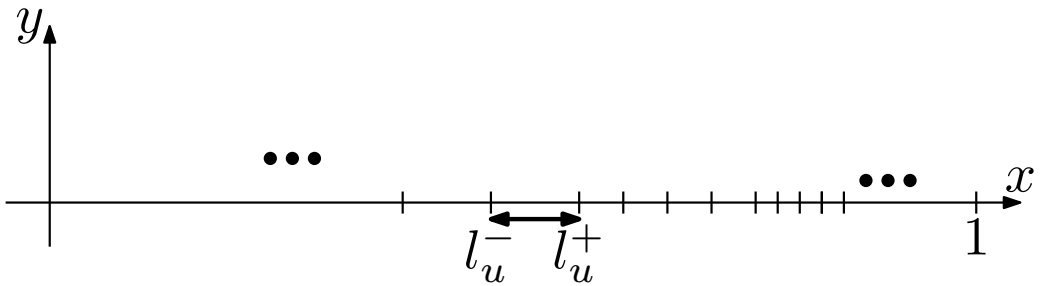

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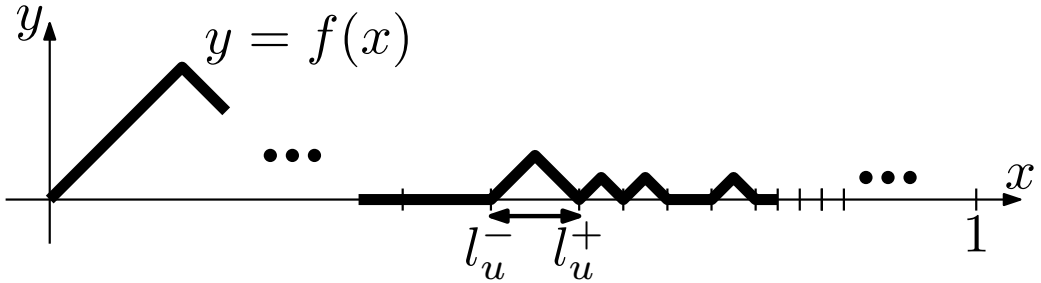

Assign interval $[l_u^-, l_u^+]$ (of length $\approx 2^{-\Theta(|u|)}$) for each $u \in \Sigma^*$. Make a bump iff $u \in A$.

Then f_A is polytime iff $A \in \mathbf{P}$.

Outline

$$h(0) = 0, \qquad h'(t) = g\big(t, h(t)\big)$$

1. Some complexity classes
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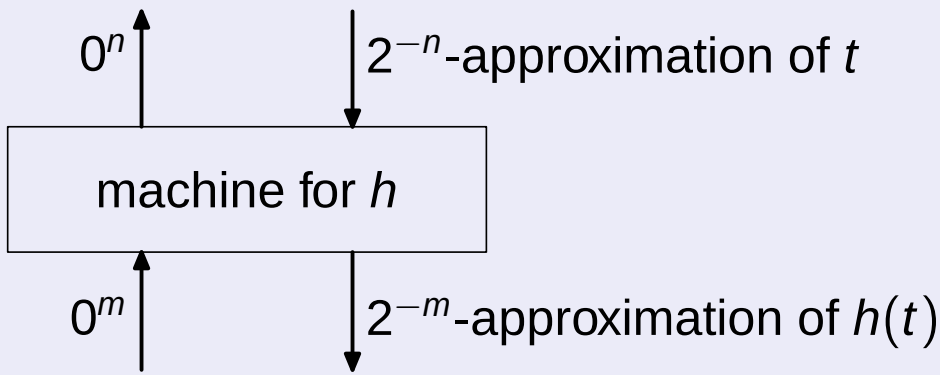

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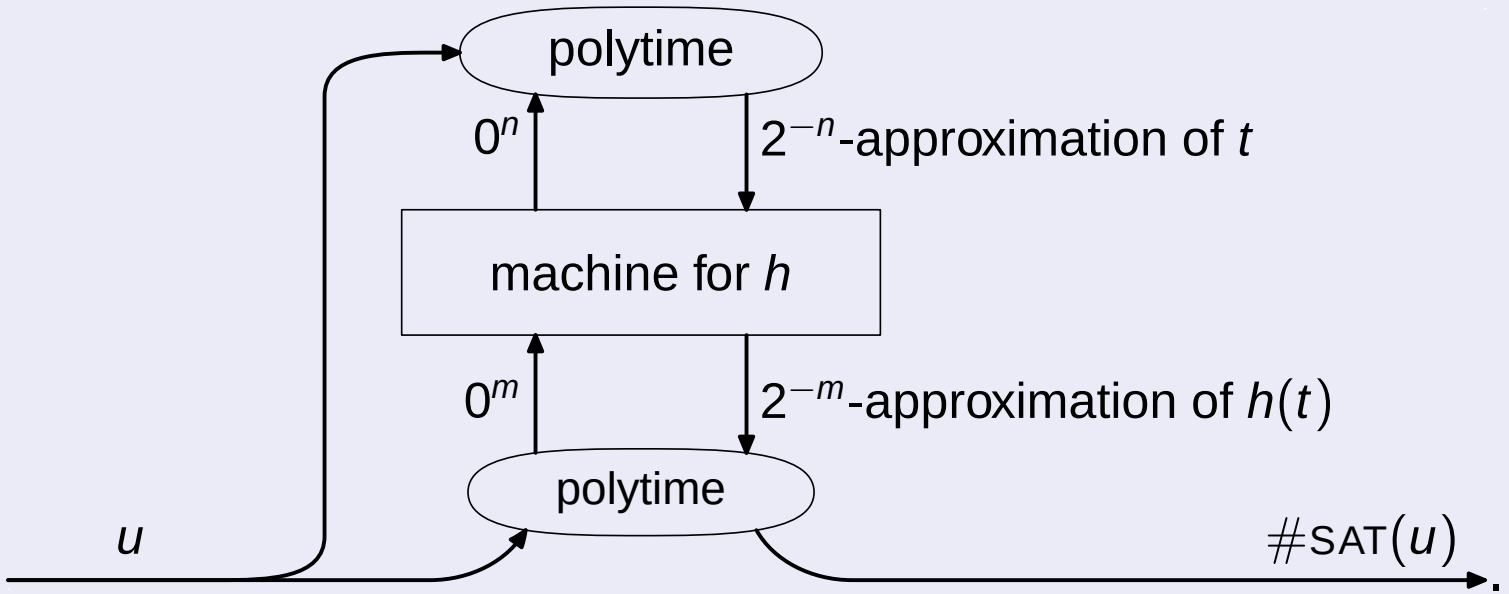
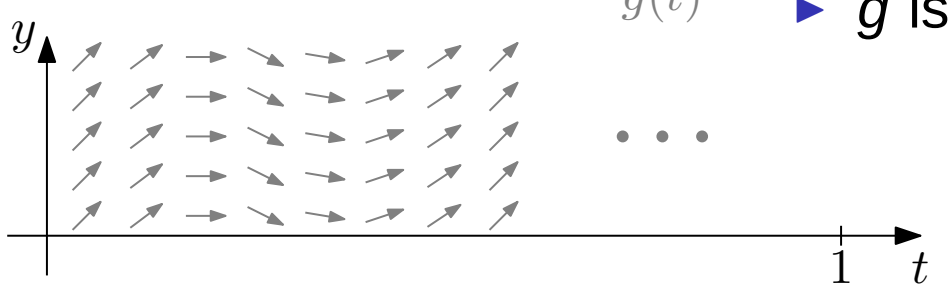


$\#\mathrm{SAT}(u)$ = number of satisfying assignments of formula u

Reducing #SAT to integration



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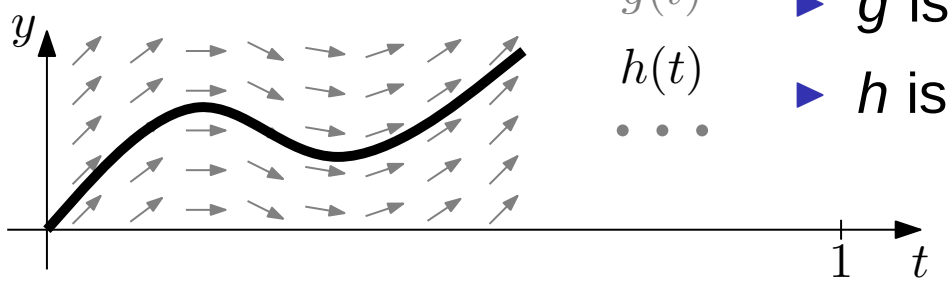

$g(t)$

$h(t)$

• • •

What we want:

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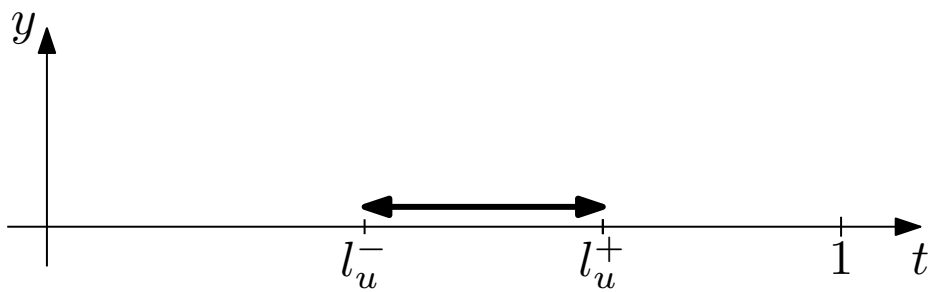

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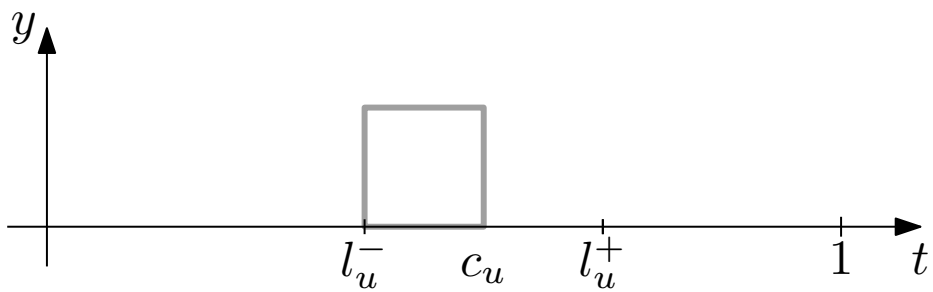

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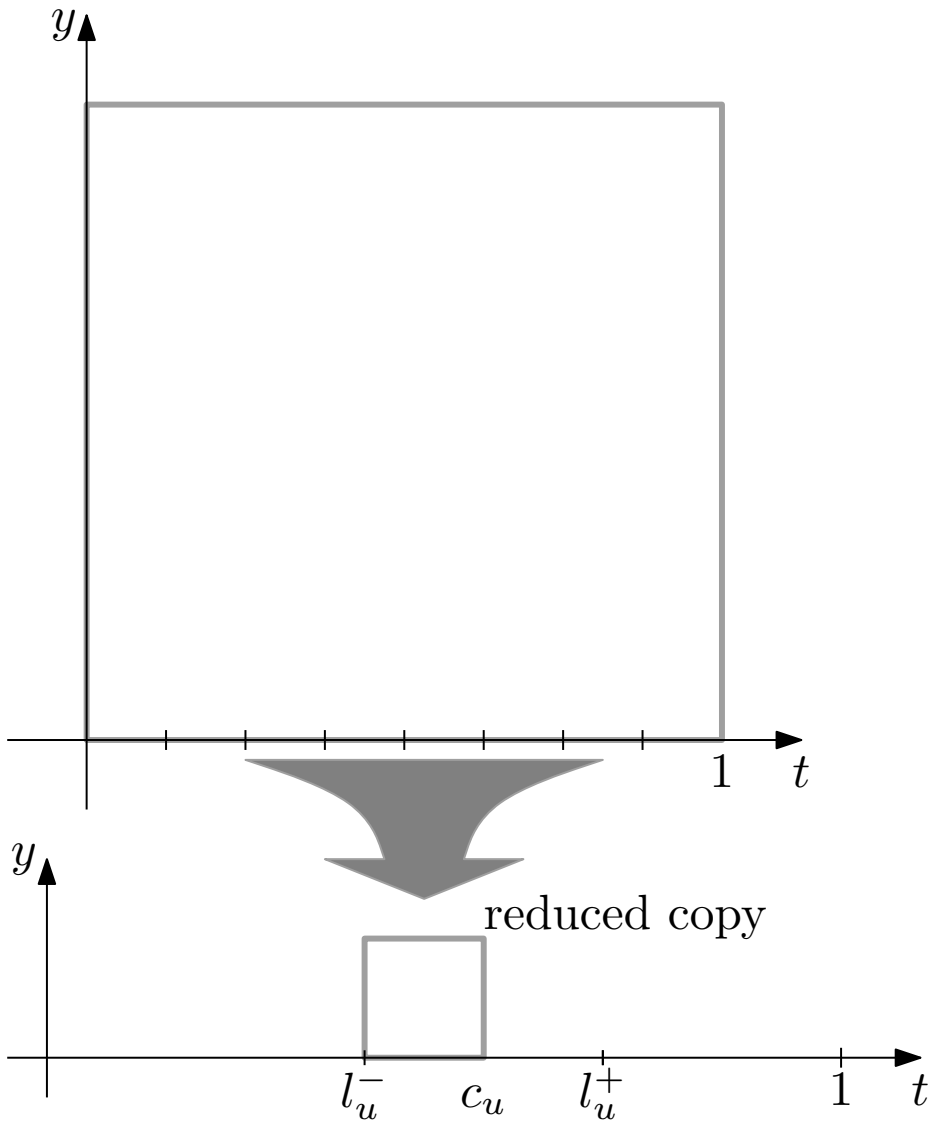

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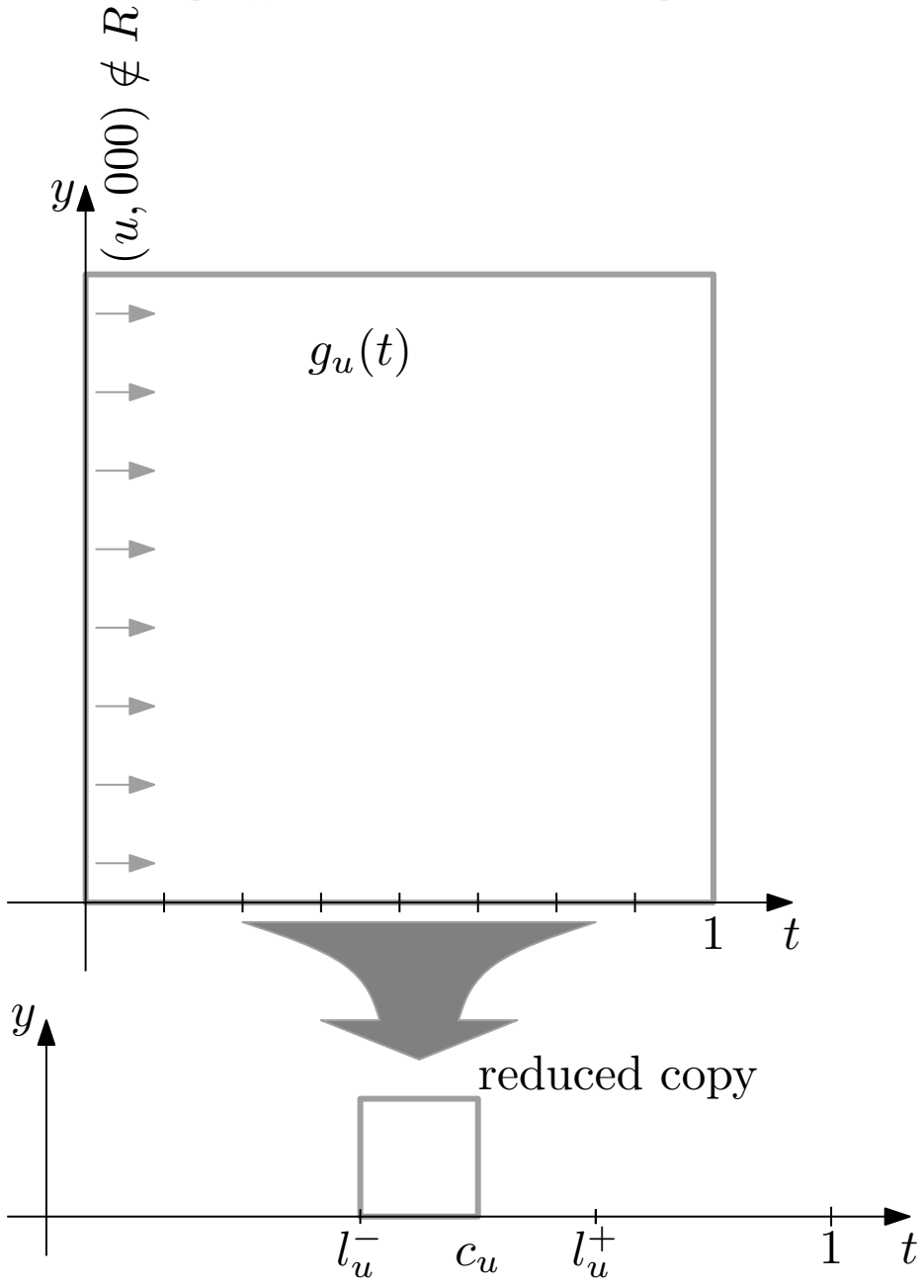

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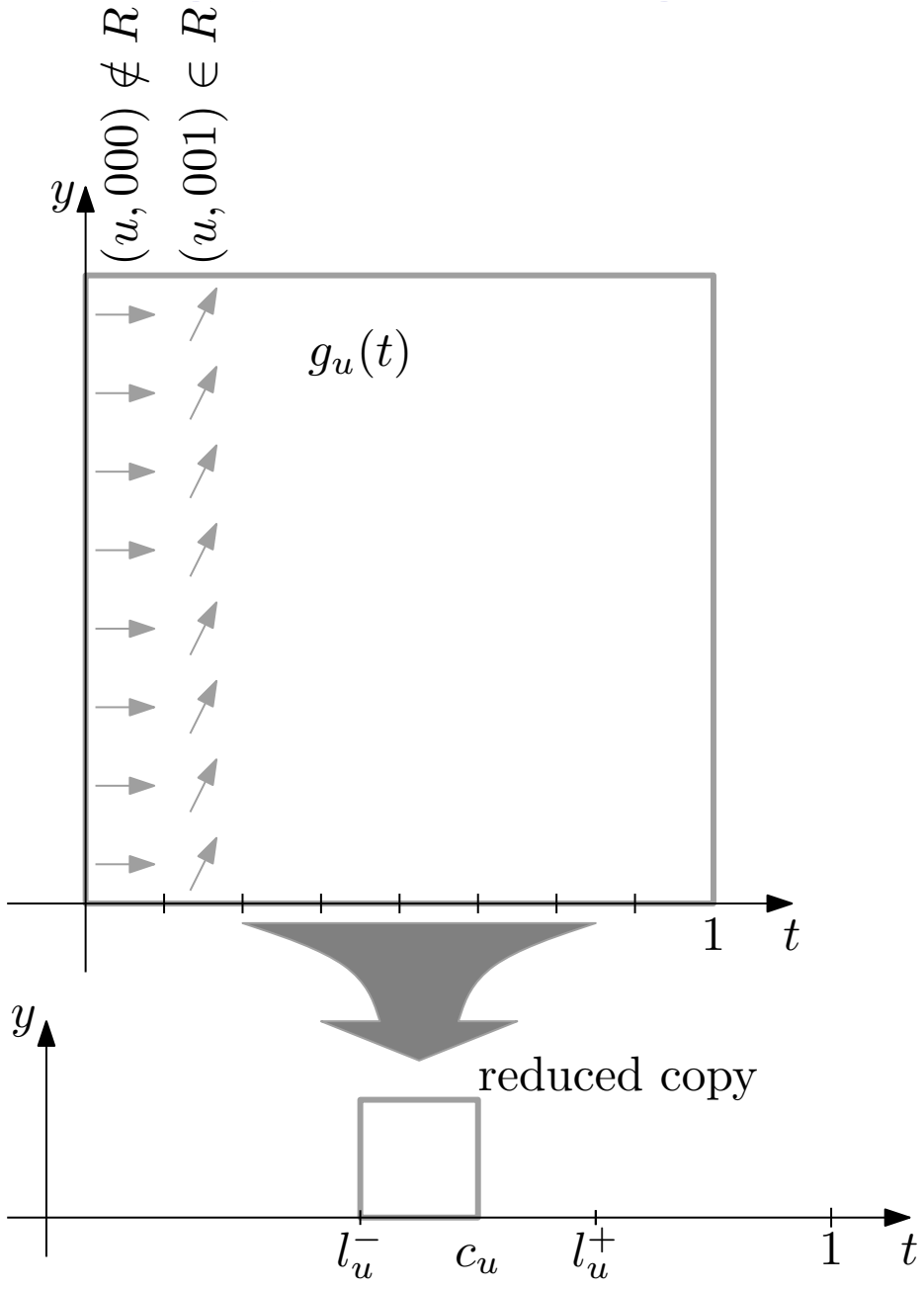

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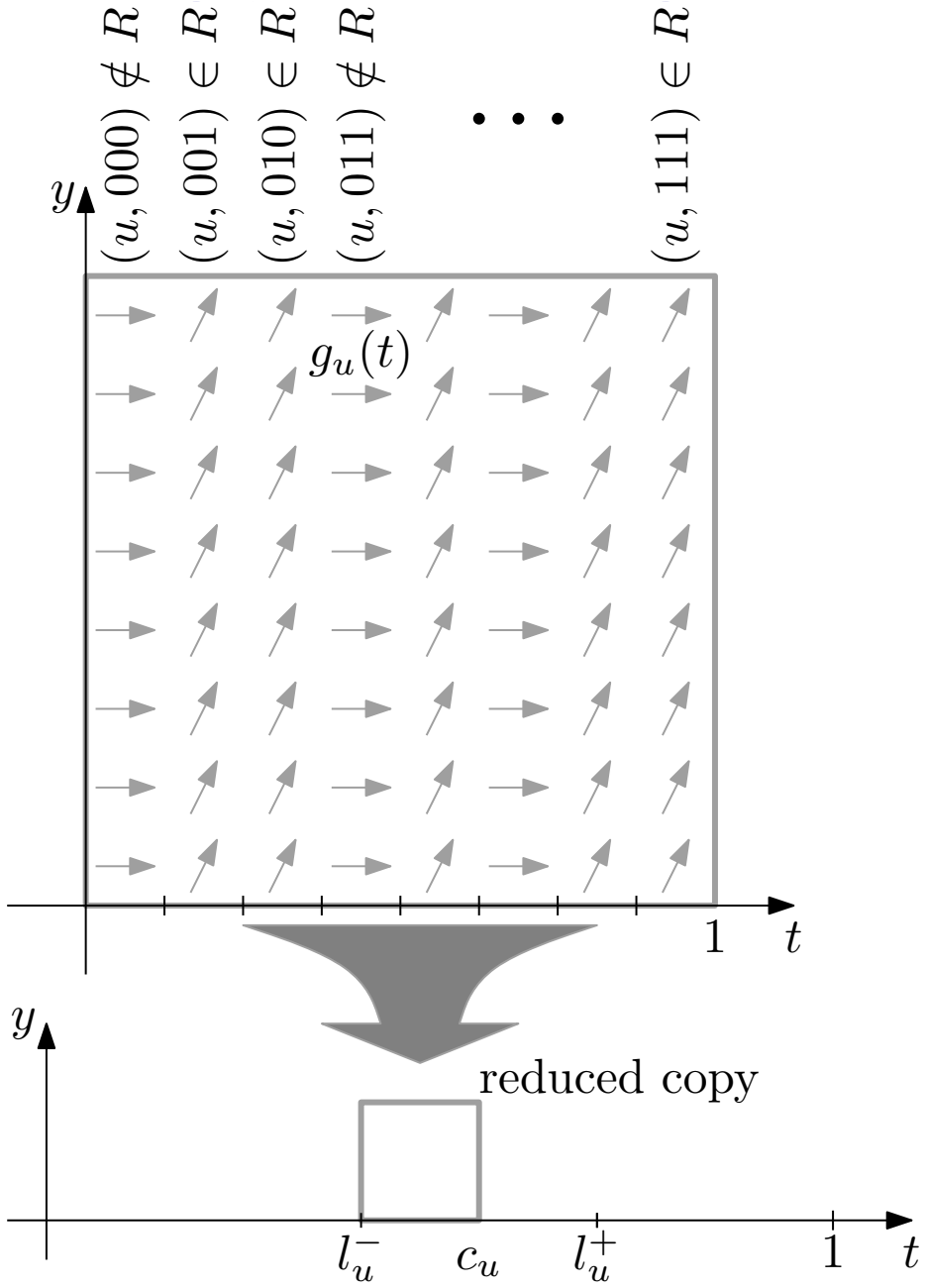

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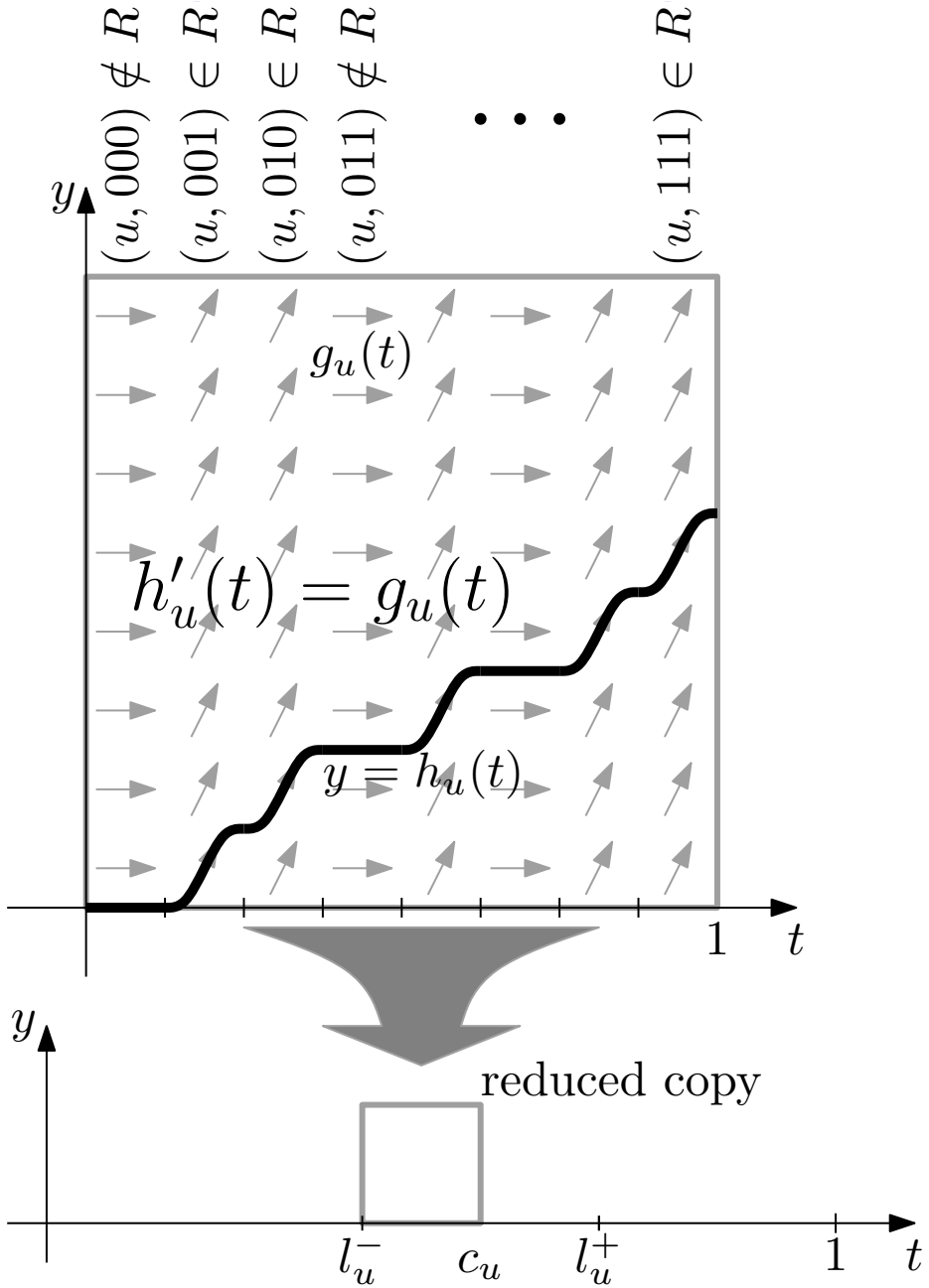

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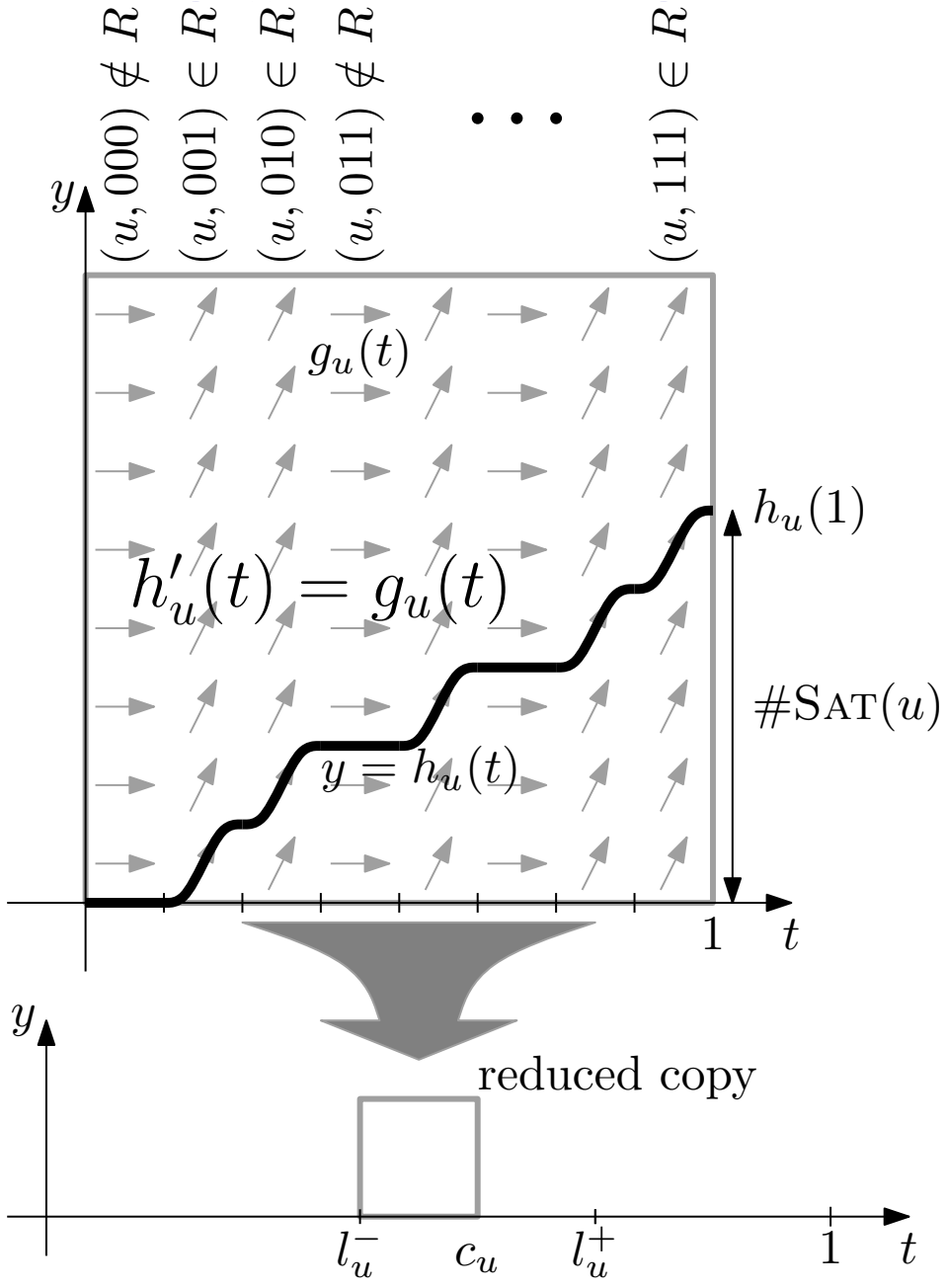

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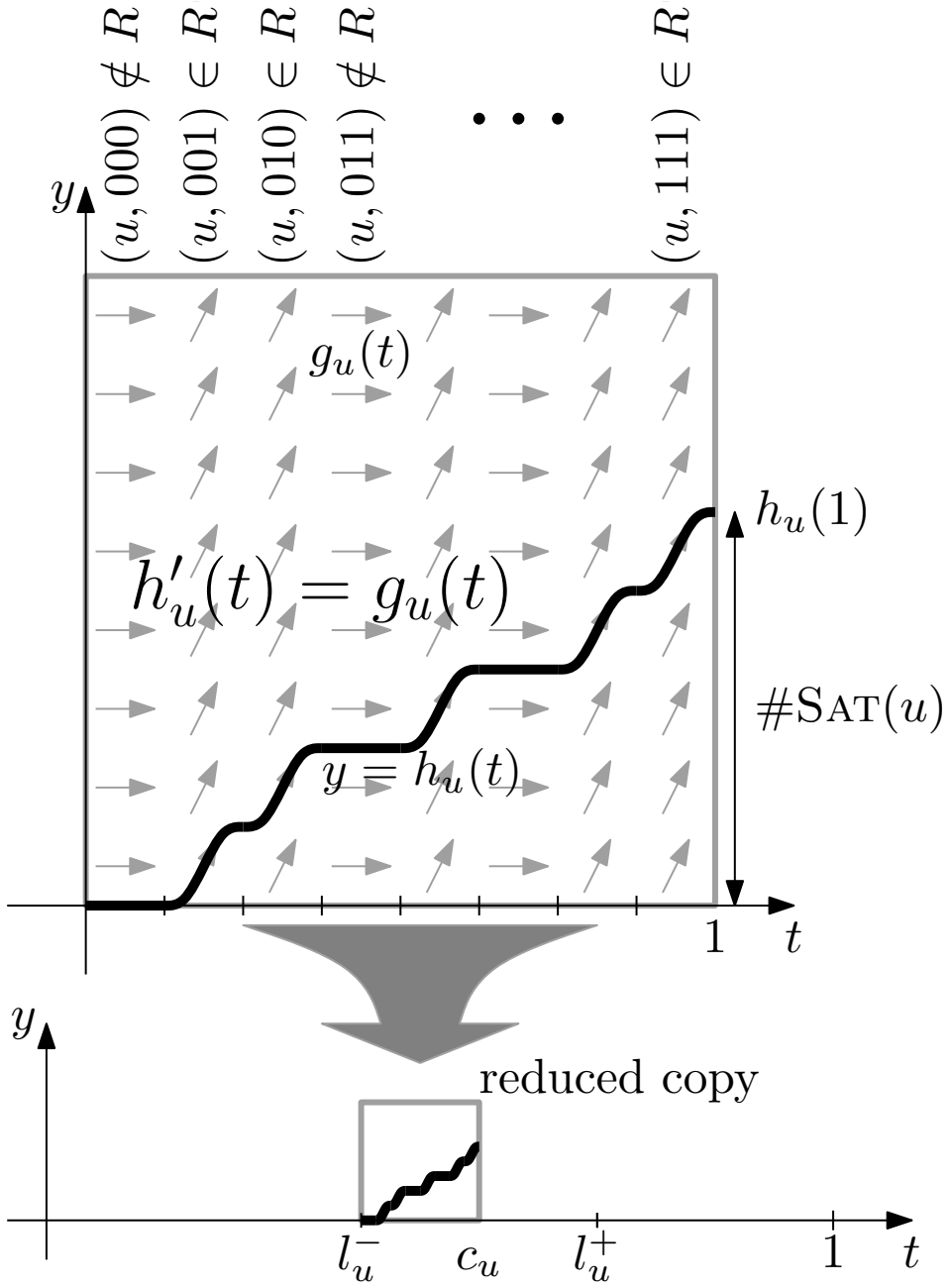

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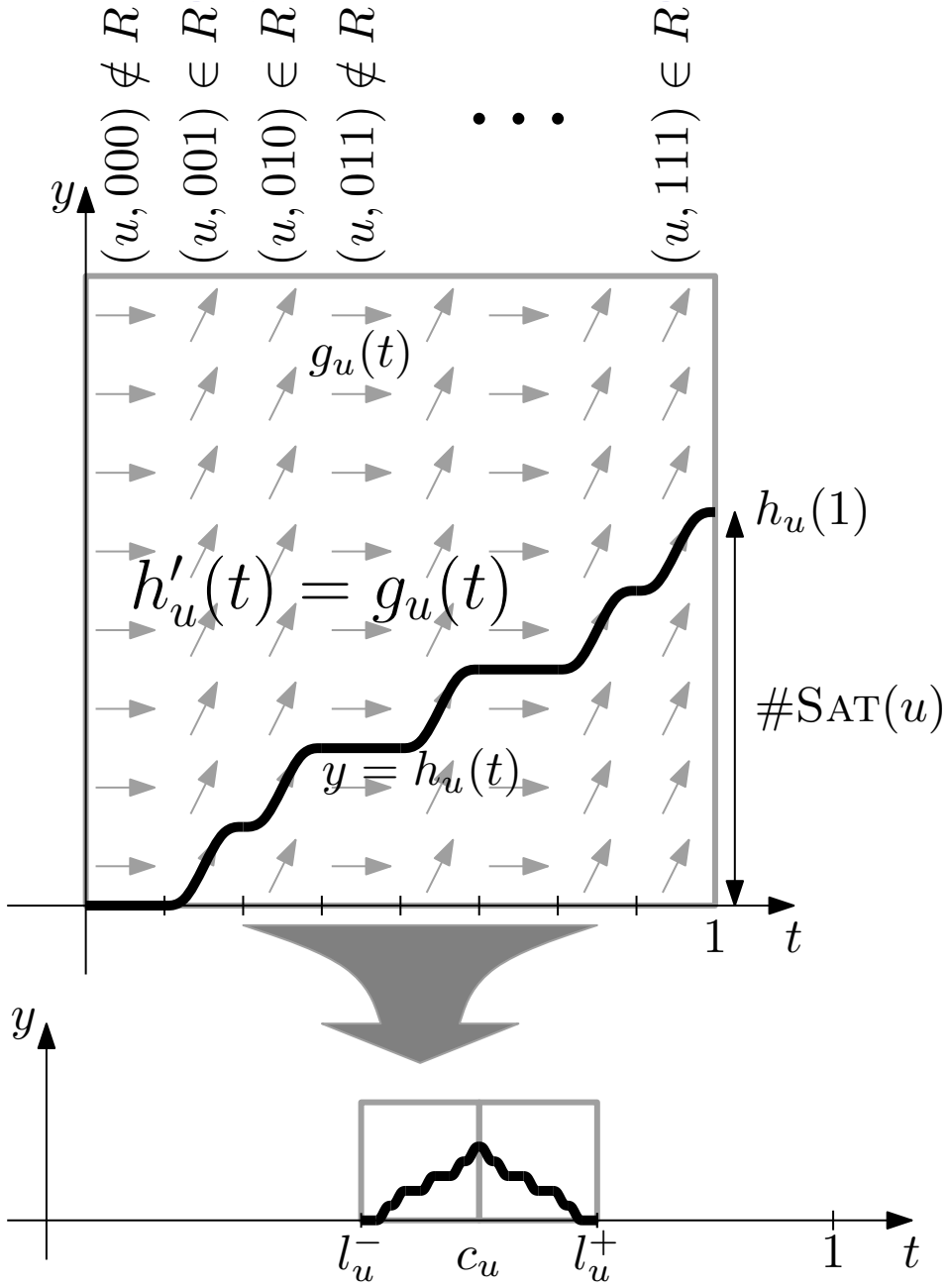

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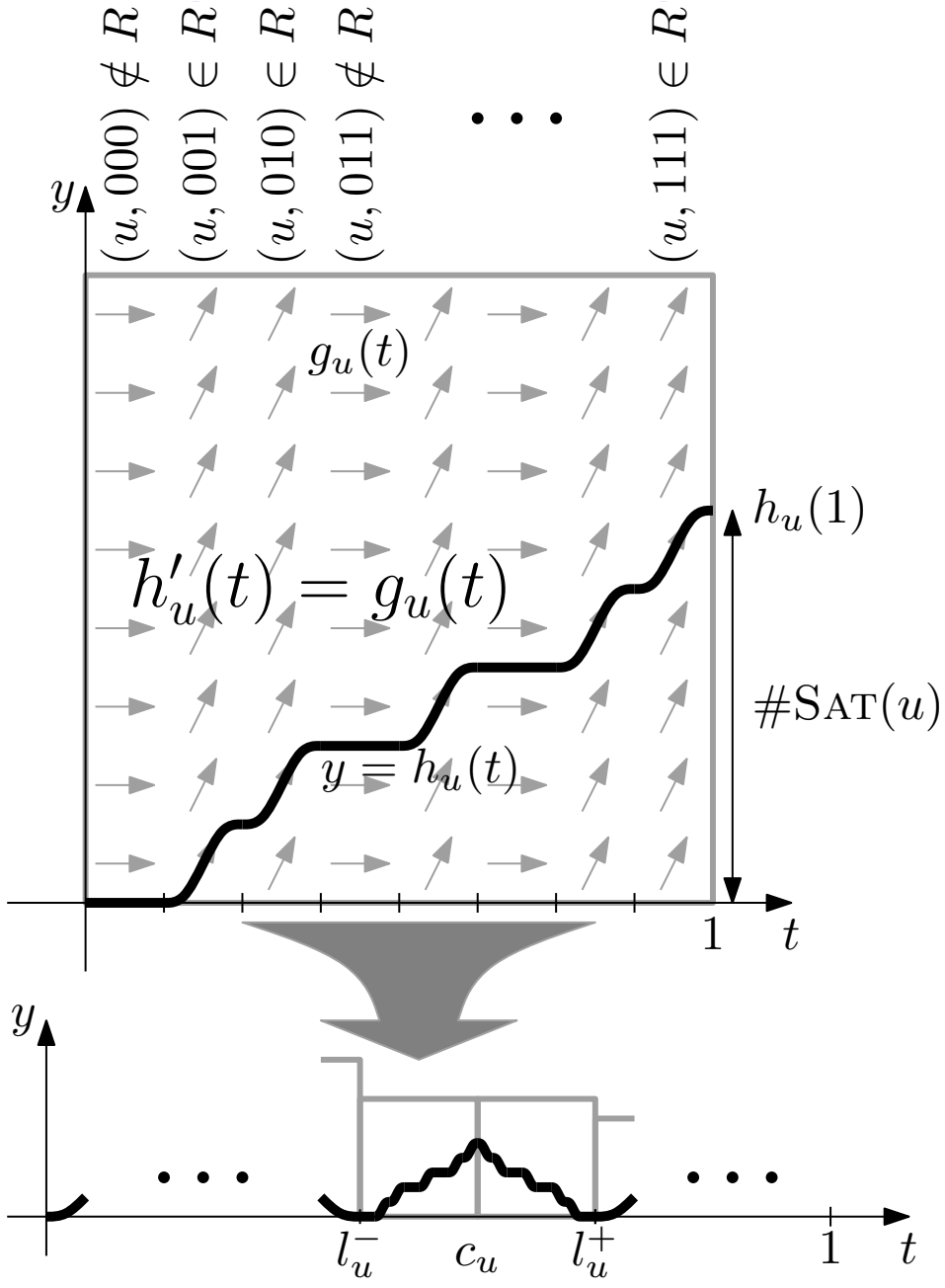

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$$h(0) = 0, \qquad h'(t) = g\big(t, h(t)\big)$$

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The Lipschitz condition

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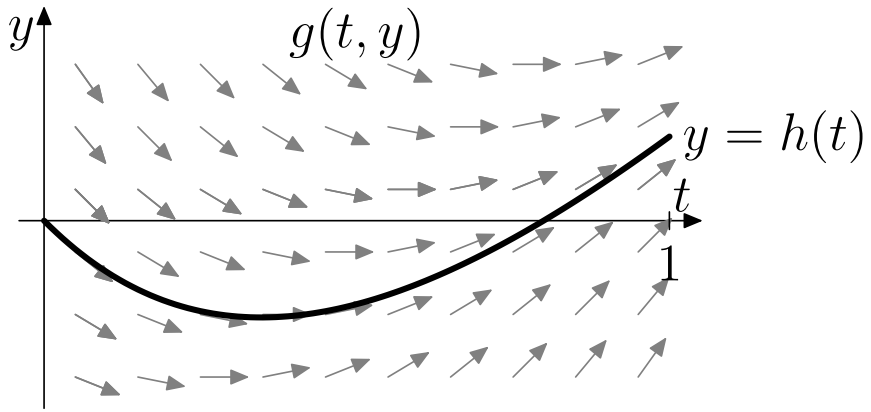

A sufficient condition to have a unique solution h is that g is Lipschitz continuous: there is some constant L such that

$$|g(t, y_0) - g(t, y_1)| \leq L|y_0 - y_1|.$$

Complexity of Lipschitz ODEs

Theorem [K 2010]

There are $g\colon [0,1]\times[-1,1]\to \mathbf{R}$ and $h\colon [0,1]\to[-1,1]$ such that

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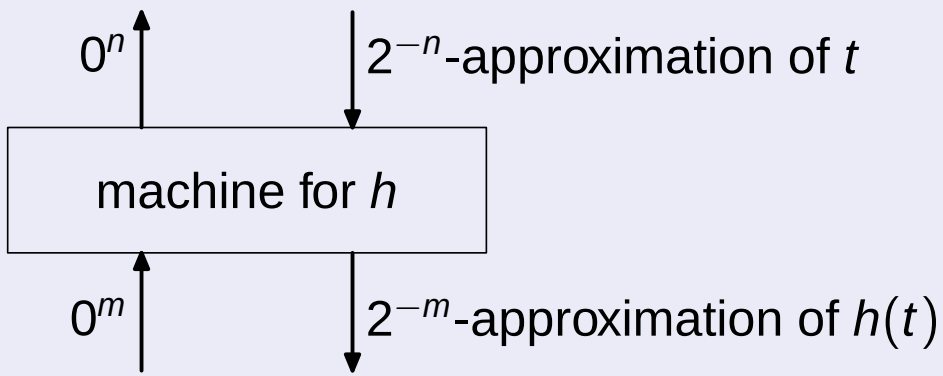

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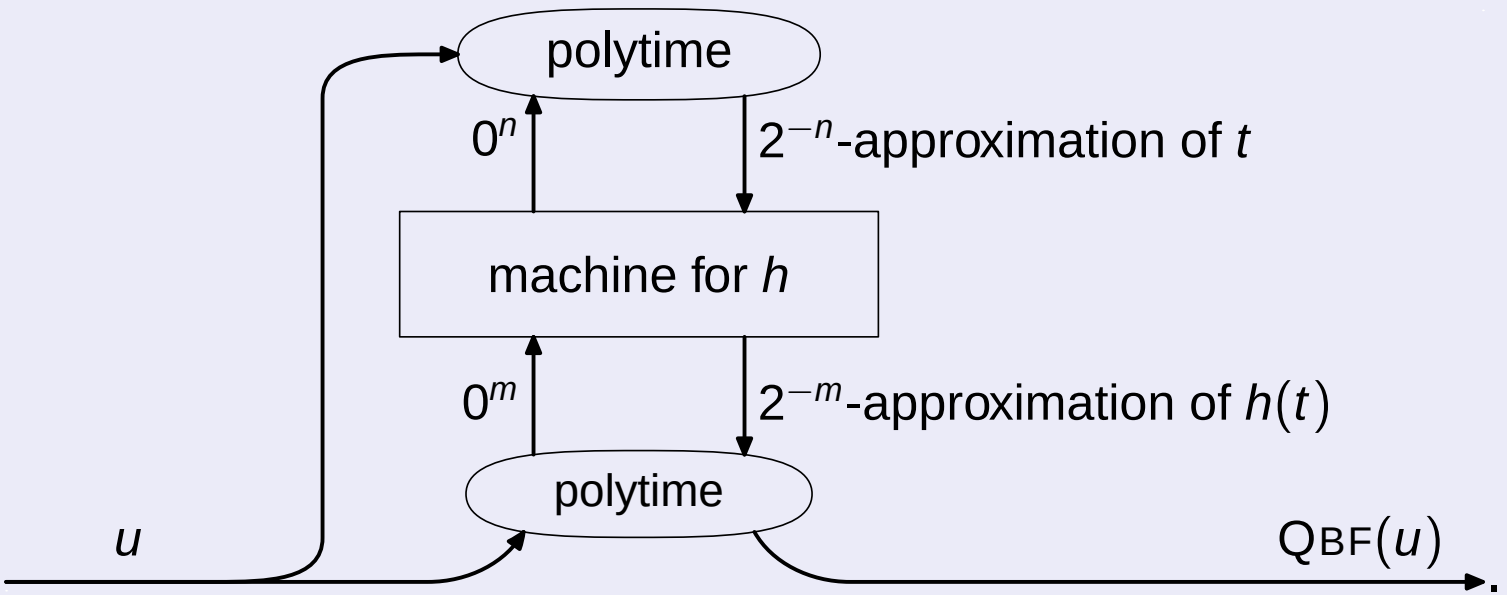

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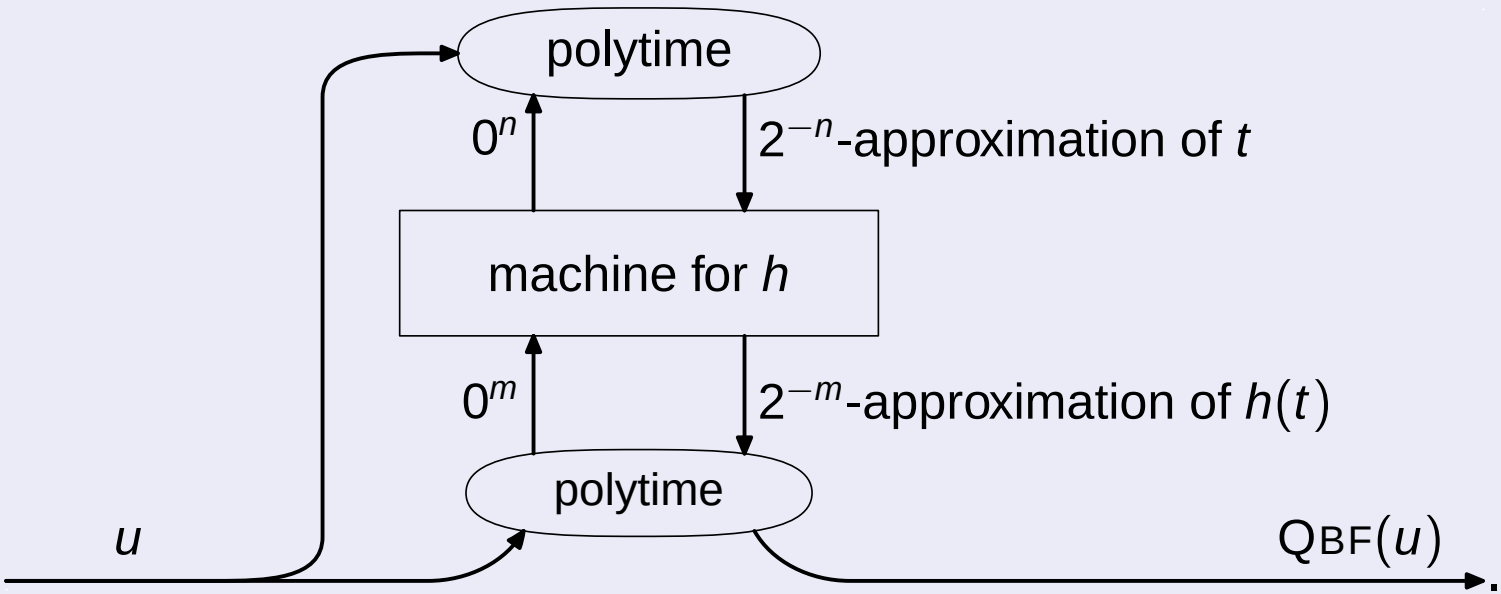
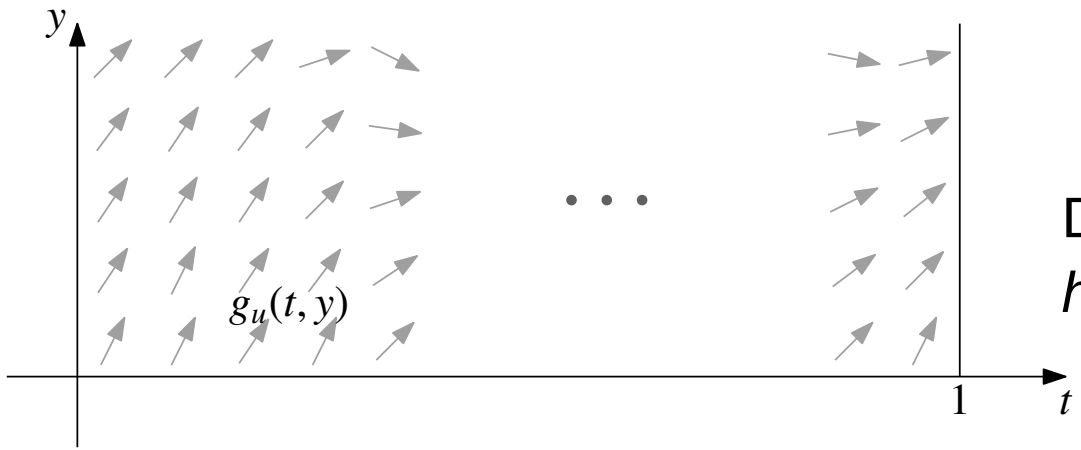


Cf. Upper bound: h is polyspace by the Euler method [Ko 1983].

Proof (1/4): An attempt to reduce **PSPACE** to ODE

As before, we need blocks g_u such that $h_u(1)$ indicates if $u \in \text{QBF}$.



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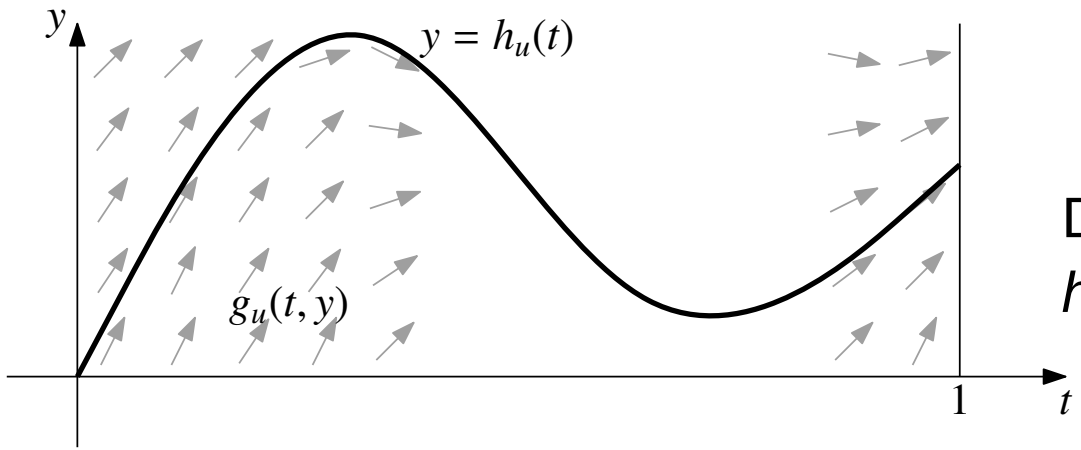

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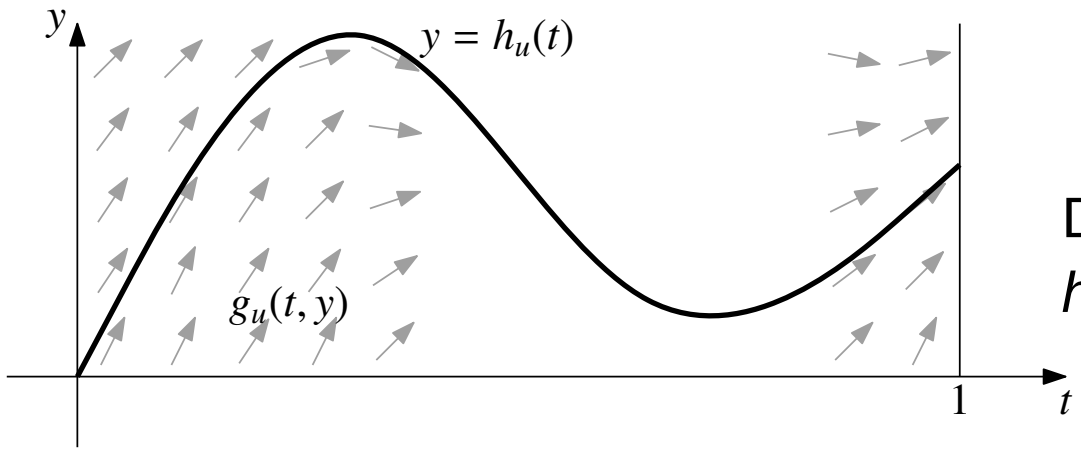

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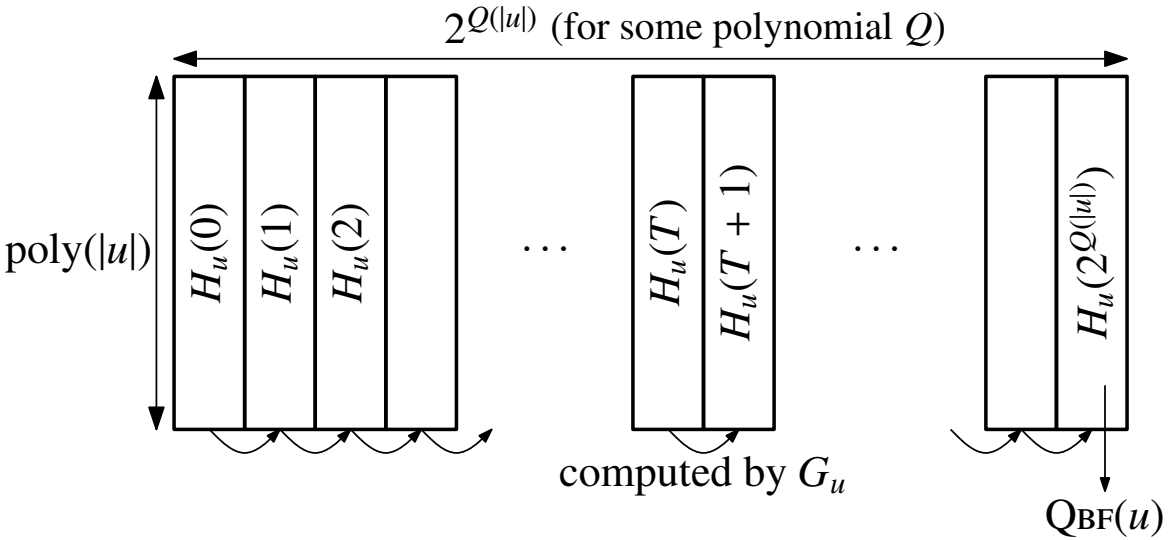

Description of a **PSPACE** machine on input u:
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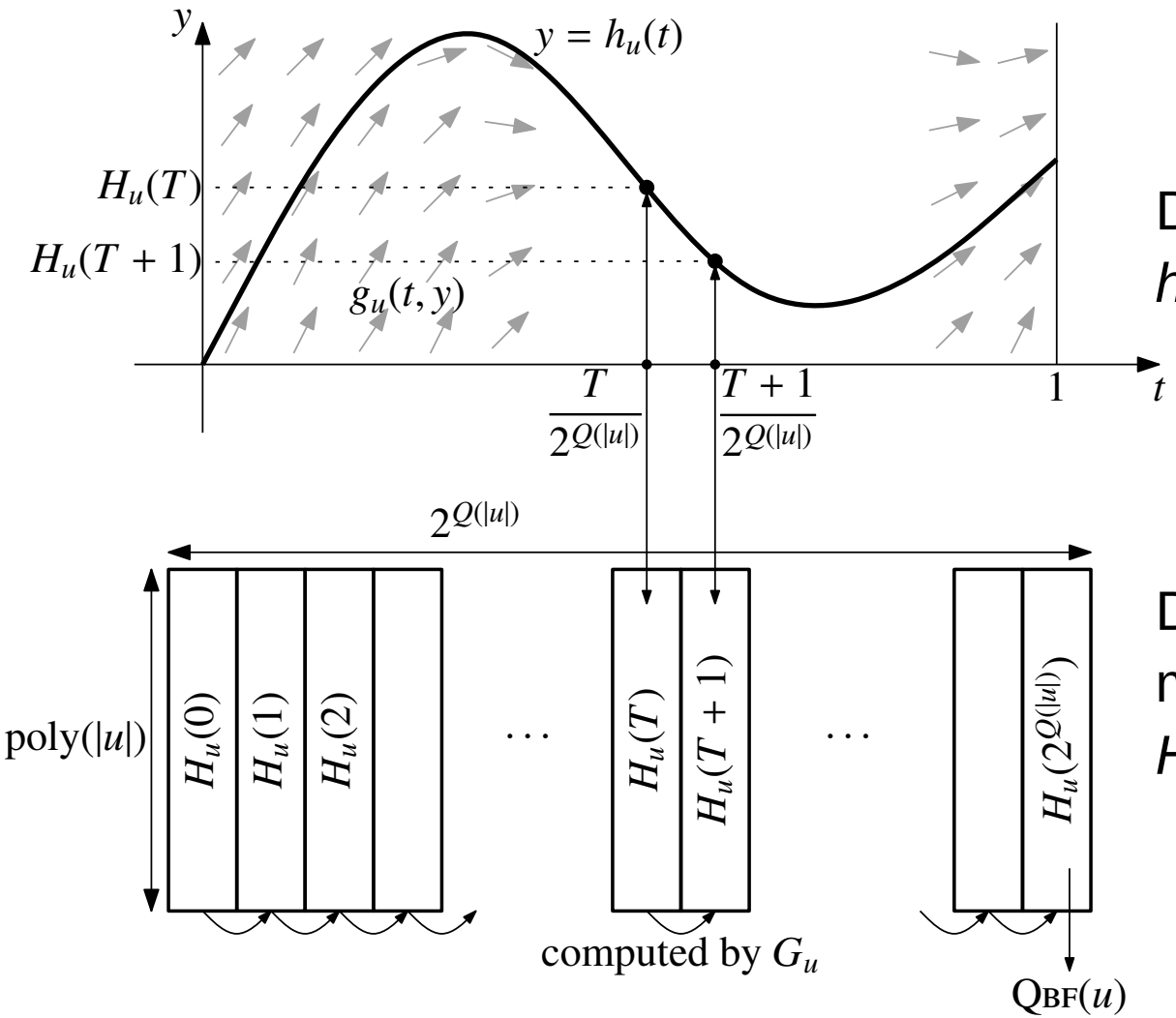

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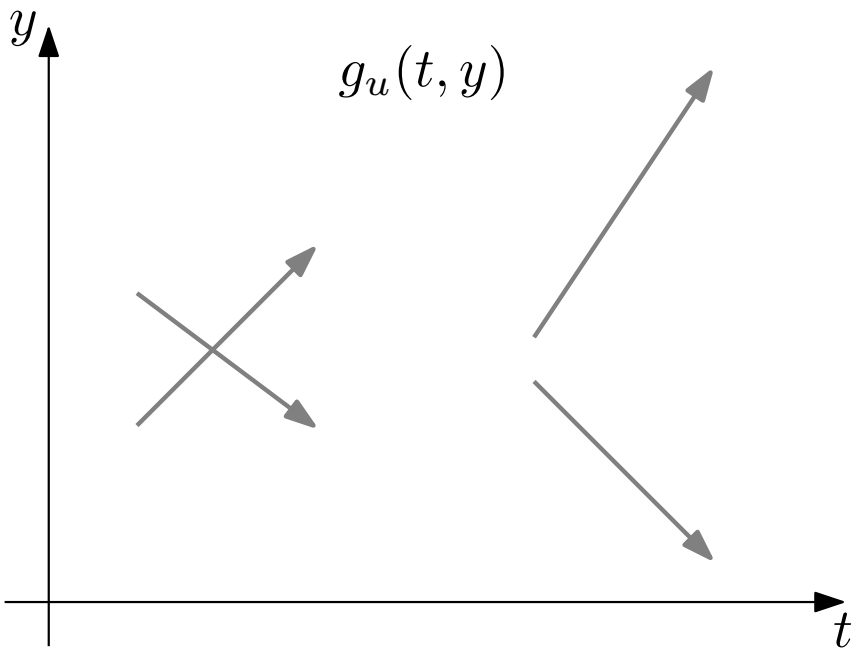
Proof (2/4): Why this attempt does not work

Not all G_u can be translated to g_u.



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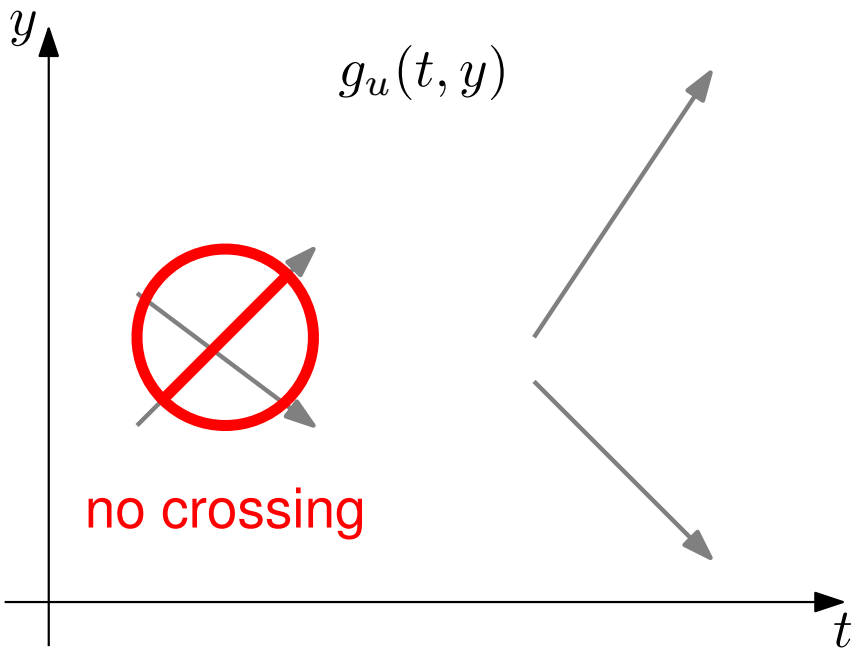

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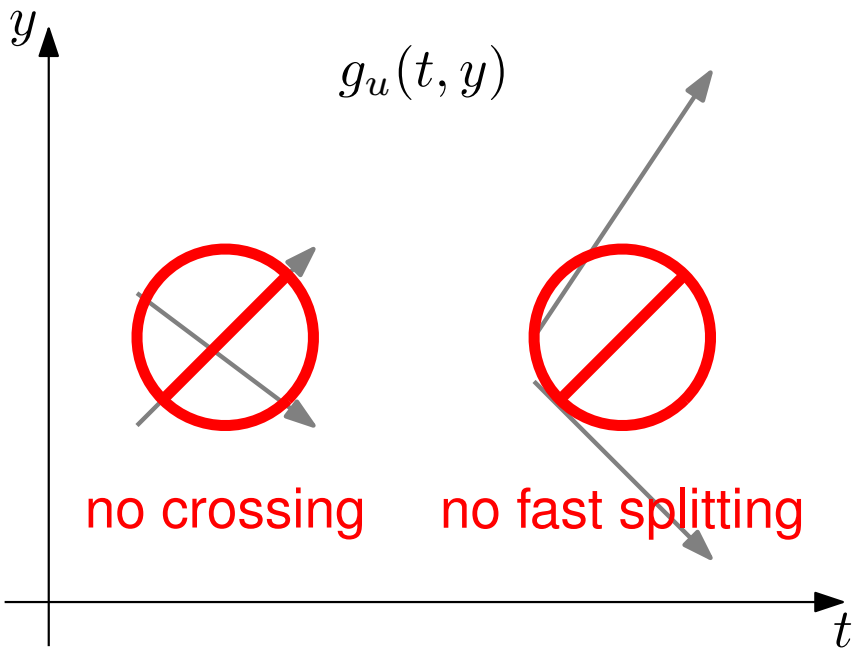

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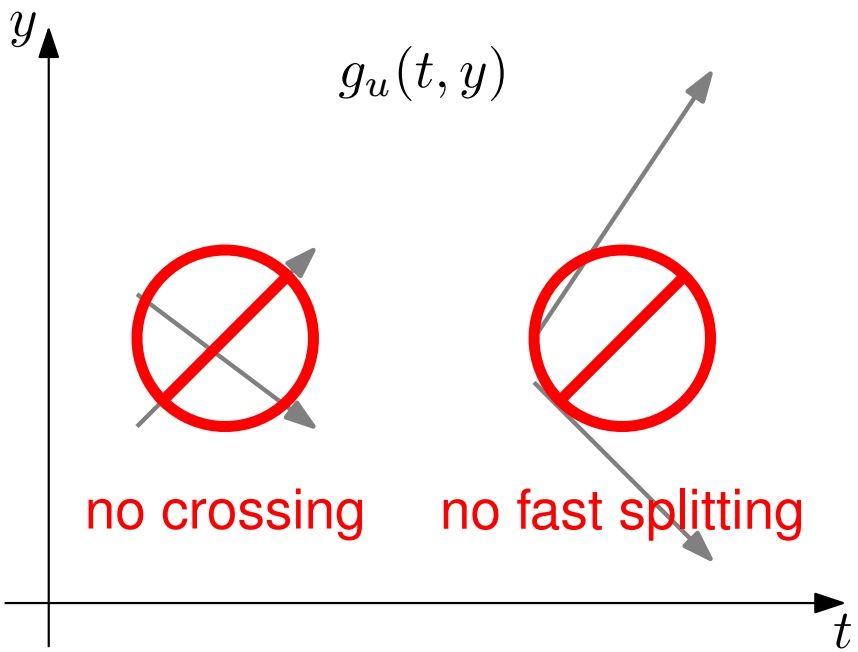

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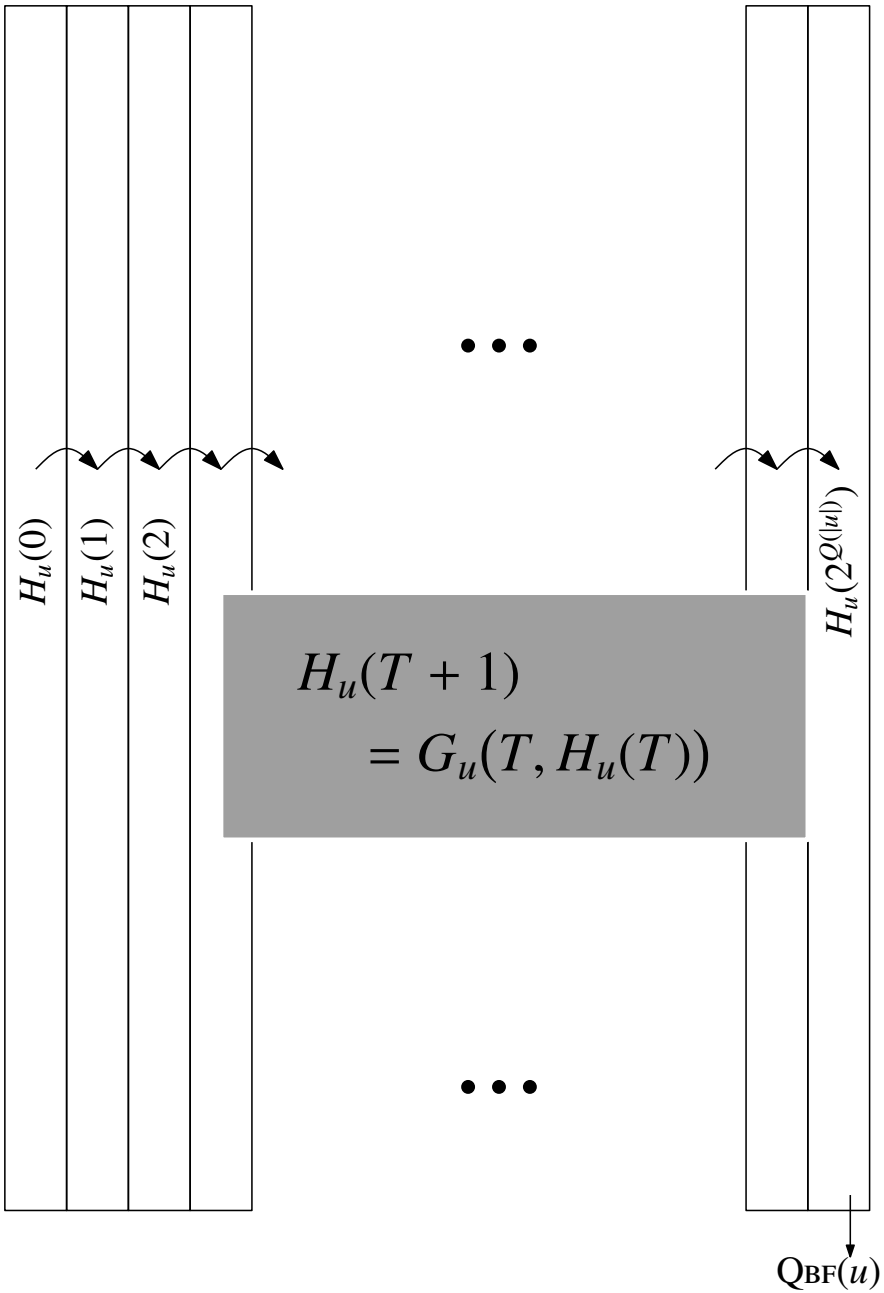
In fact, the Lipschitz condition forbids the flows from widening or narrowing fast.

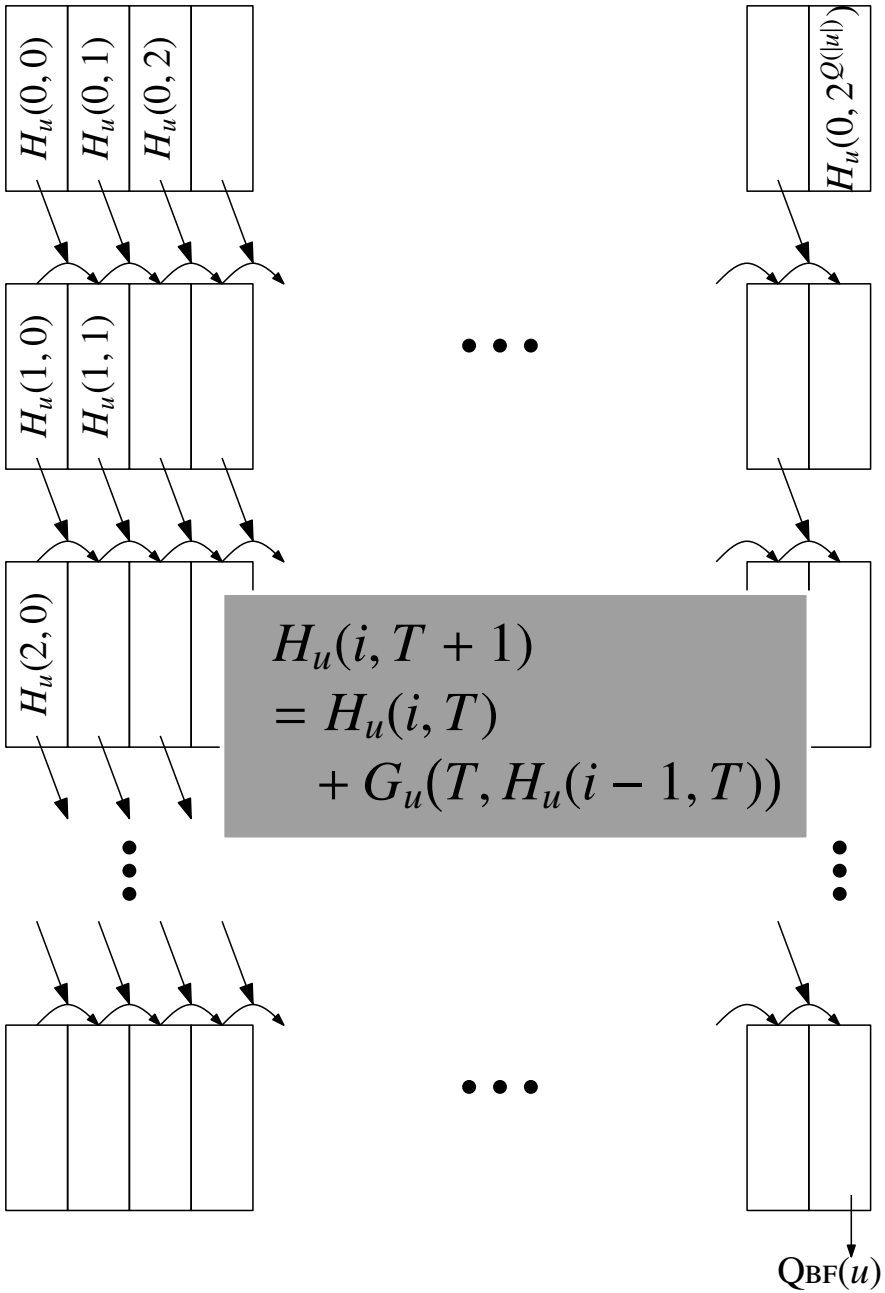
Thus, the 'feedback' (on h_u) described by the equation $h'_u(t) = g_u\big(t, h_u(t)\big)$ is very weak.

Proof (3/4): Layered **PSPACE** tableaux



Differential equations cannot simulate general **PSPACE** computation tableaux.

Proof (3/4): Layered **PSPACE** tableaux



Differential equations cannot simulate general **PSPACE** computation tableaux.

But it can simulate layered tableaux where each cell is split into parts and each part only affects parts below itself.

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Despite the restriction, the layered tableau is **PSPACE**-complete, by a reduction from QBF.

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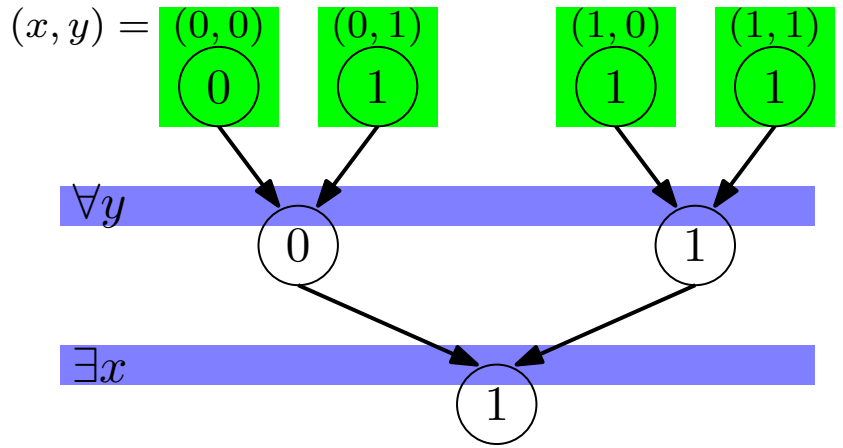

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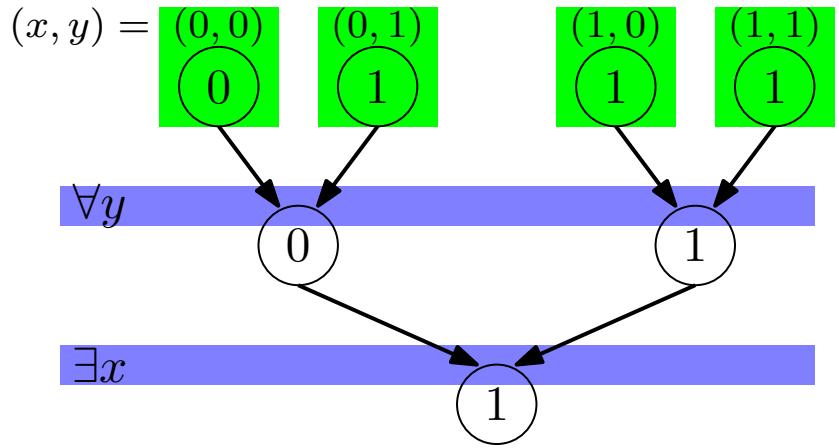

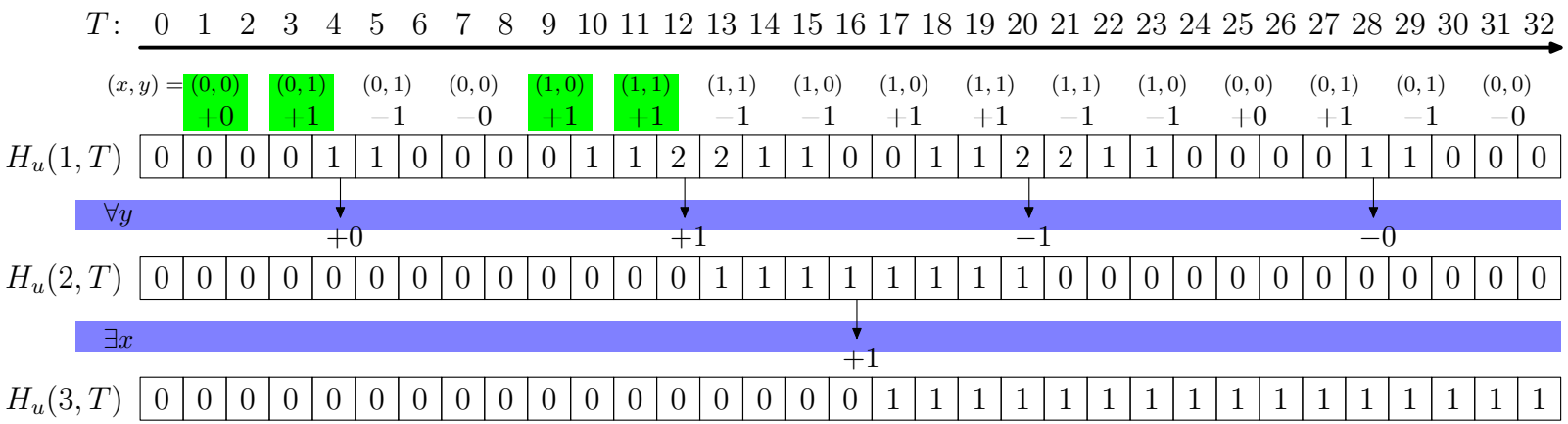

Complexity of smooth ODEs

Theorem [K, Ota, Rösnick, Ziegler 2012]

For any fixed $k \in \mathbf{N}$, there are g and h such that

- g is polytime and of class C^k;
- $h(0) = 0, \quad h'(t) = g\big(t, h(t)\big)$;
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Similar argument, but with only constant number of layers.

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Similar argument, but with only constant number of layers.

Open:

- Close the gap between this and the polyspace upper bound.
- C^∞ functions?
- Does smoothness generally help, for other operators?
  - Not for integration (still #**P**-hard for C^∞ functions)
  - nor for maximization (still **NP**-hard for C^∞ functions)

Outline

$$h(0) = 0, \qquad h'(t) = g\big(t, h(t)\big)$$

1. Some complexity classes
   P, **NP**, **#P**, **PSPACE**, ...
2. Complexity of real functions
   How we define polytime real functions
3. Warm-up: Integration
   Assuming g is polytime and ignores the second argument, how complex can h be?
4. Lipschitz continuous ODE
   Assuming g is polytime and Lipschitz continuous, how complex can h be?
5. ➤ Final remarks
   Uniform versions / Summary

Non-uniform and uniform statements

Our theorem stated:

Theorem [K 2010]

There is $g : [0,1] \times [-1,1] \to \mathbf{R}$ such that

- g is polytime and Lipschitz continuous;
- $Solve_{\mathrm{Lip}}(g)$ is **PSPACE**-hard.

where $Solve_{\mathrm{Lip}}$ is the operator mapping g to the solution h.

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But can we say something like:

Theorem (?)

$Solve_{\mathrm{Lip}}$ is **PSPACE**-hard.

Uniform version (1/4): Encoding

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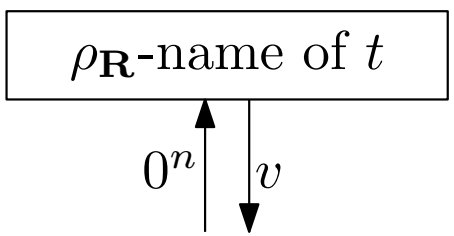
Polytime real functions were defined by a polytime oracle machine and an encoding $\rho_{\mathbf{R}}$ of real numbers by string functions.
(Henceforth we will say $(\rho_{[0,1]}, \rho_{\mathbf{R}})$-polytime, etc.)



$$|v - t| \leq 2^{-n}$$

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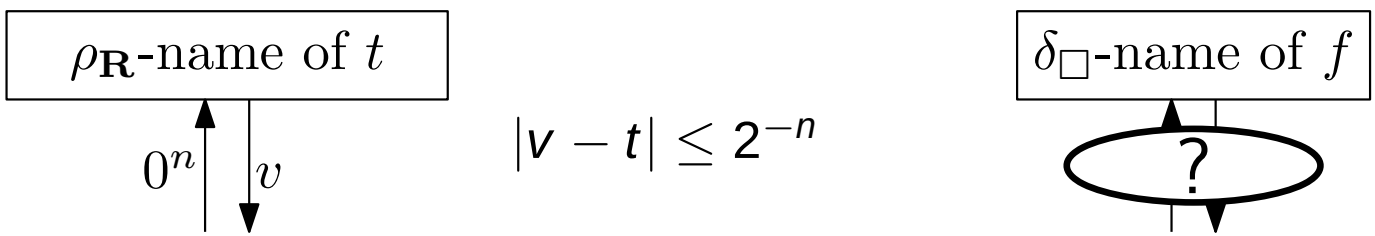

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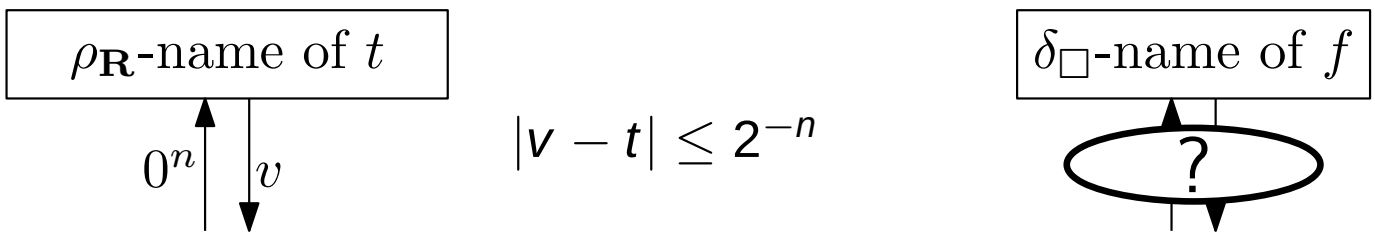

To make this work,
we need to use oracles whose answers are unbounded in length and extend the notion of polytime accordingly [K and Cook 2012].

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Encoding $\delta_\square$ of continuous functions $f : [0,1] \to \mathbf{R}$:

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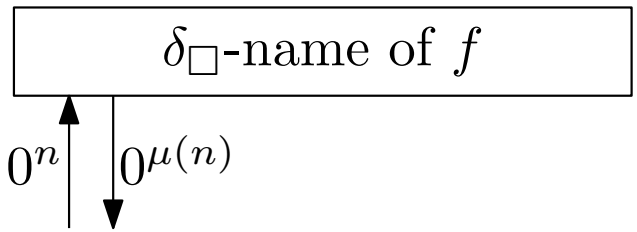

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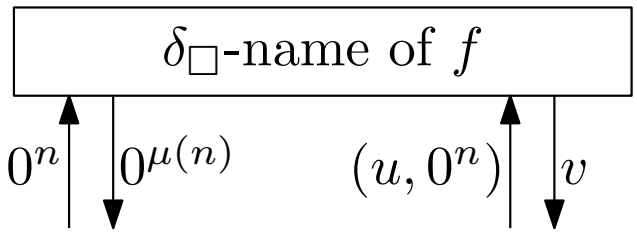

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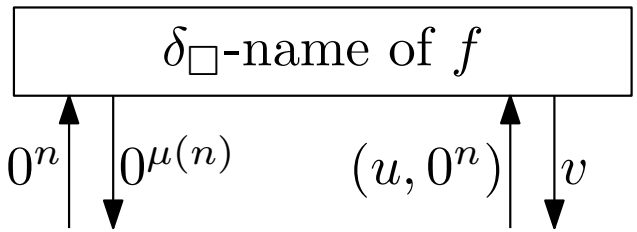

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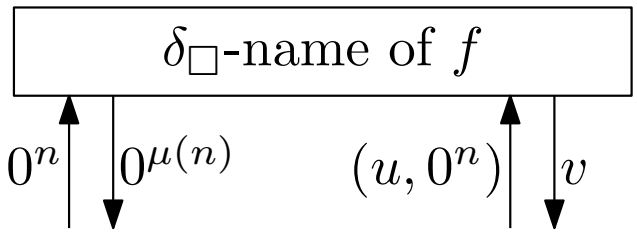

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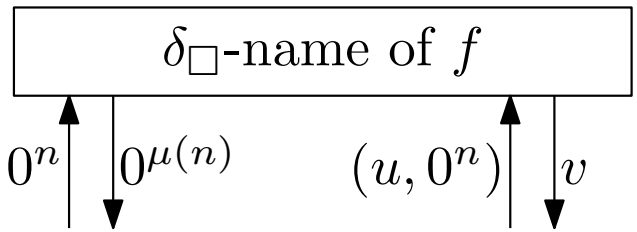

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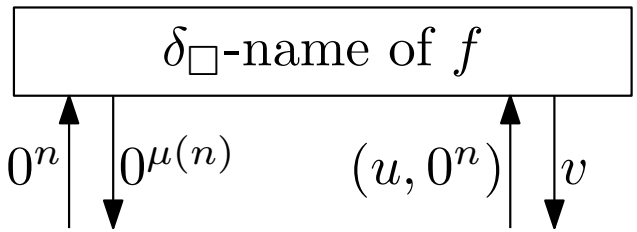

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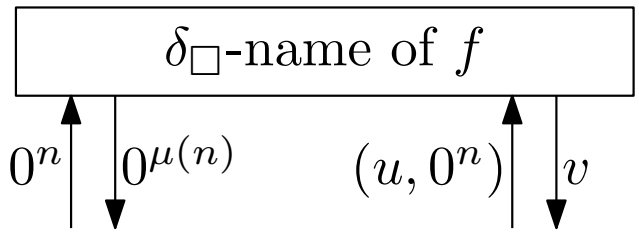

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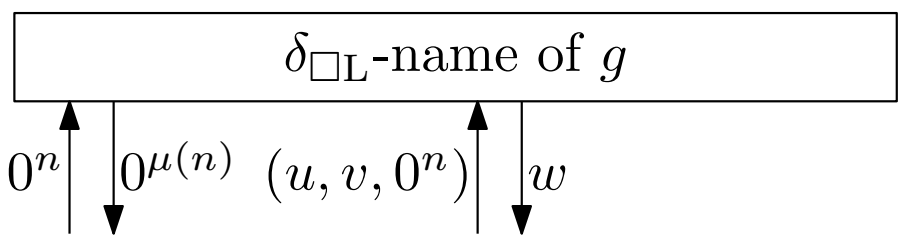

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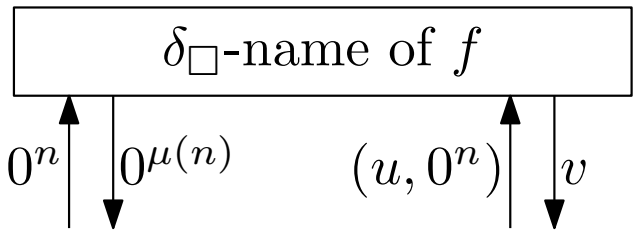

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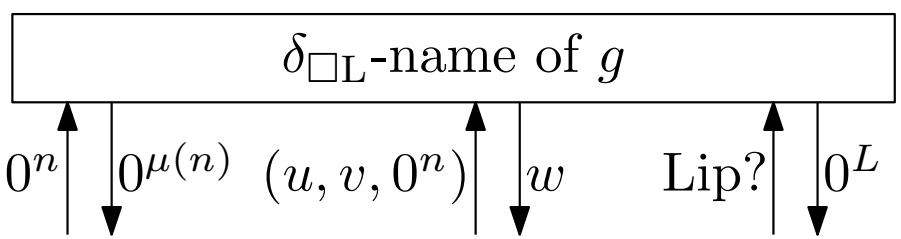

μ: modulus of continuity of g
$|w - g(u,v)| \leq 2^{-n}$
$L \in \mathbf{N}$: Lipschitz constant of g

Uniform version (3/4): Lipschitz ODE

By relativizing the proof, we can show:

Theorem (uniform)

- $\mathit{Solve}_{\mathrm{Lip}}$ is $(\delta_{\square \mathrm{L}}, \delta_{\square})$-polyspace.
- $\mathit{Solve}_{\mathrm{Lip}}$ is $(\delta_{\square \mathrm{L}}, \delta_{\square})$-polyspace-hard (wrt a suitable reduction).

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Corollary (non-uniform)

- If g is $(\rho_{[0,1]}, \rho_{\mathbf{R}})$-polyspace, so is $Solve_{\mathrm{Lip}}(g)$.
- There is a $(\rho_{[0,1]}, \rho_{\mathbf{R}})$-polytime function g such that $Solve_{\mathrm{Lip}}(g)$ is $(\rho_{[0,1]}, \rho_{\mathbf{R}})$-polyspace-hard.

Uniform version (4/4): Analytic ODE

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If g is $(\rho_{[0,1]}, \rho_{\mathbf{R}})$-polytime, so is $Solve_{\text{Analytic}}(g)$.

Uniform version (4/4): Analytic ODE

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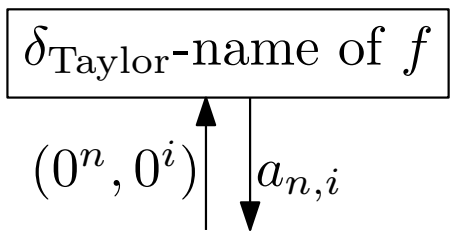
$Solve_{\text{Analytic}}$ is $(\delta_{\text{Taylor}}, \delta_{\text{Taylor}})$-polytime.



$f(t) = \sum_{i \in \mathbf{N}} a_i x^i$ for t near 0
$|a_{n,i} - a_i| \leq 2^{-n}$

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If g is $(\rho_{[0,1]}, \rho_{\mathbf{R}})$-polytime, so is $Solve_{\text{Analytic}}(g)$.

Summary: Complexity of ODE solutions

$$h(0) = 0, \qquad h'(t) = g\big(t, h(t)\big)$$

Assuming g is polytime, how complex can h be?

Assumptions	Upper bound	Lower bound
None	——	can be (non-unique and) all non-computable [Pour-El 1979]
h is the unique solution	computable [implicit in Osgood 1898]	can take arbitrarily long time (or space) [Miller 1970]
g is 'locally' Lipschitz [Ko 1992]	exponential space [Ko 1992]	can be **EXPSPACE**-hard [K 2010]
g is Lipschitz	polyspace [Ko 1983]	can be **PSPACE**-hard [K 2010] (CCC 2009)
g is of class C^1		can be **PSPACE**-hard [K, Ota, Rösnick, Ziegler 2012] (MFCS 2012)
g is of class C^k		can be **CH**-hard [ibid.]
g is analytic	polytime [Müller 1987]	——

These results can be made uniform, wrt different representations.